# Game Theory

# for

# Wireless Networks

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# OBJECTIVES

•Introduce non-cooperative game notions that are potentially useful in wireless networking. **1st part: Introduction, matrix games.**

1. Applications of games in networking
2. Non-cooperative games
3. Zero-sum games
4. Correlated equilibrium
5. Constrained games

# OBJECTIVES

•Introduce non-cooperative game notions that are potentially useful in wireless networking:

1. Non-atomic games
2. Wardrop equilibrium,
3. Potential games for infinite player set
4. Replicator dynamics
5. Discrete Potential games, Convergence
6. Concave games (Rosen)
7. Constrained games and normalized equilibrium
8. S-modular games

# 1 Networking Games Examples and Classification

## 1.1 The Association Probelem

• To which WIFI shall we connect?

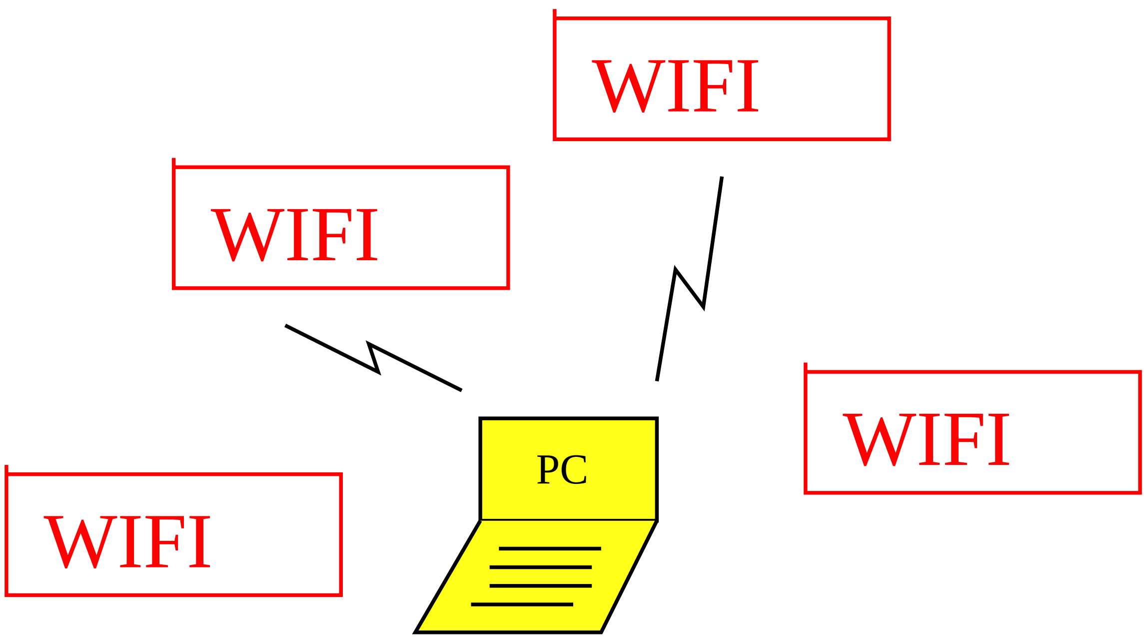



Figure 1: The Association Problem (1)

# The man-machine interface

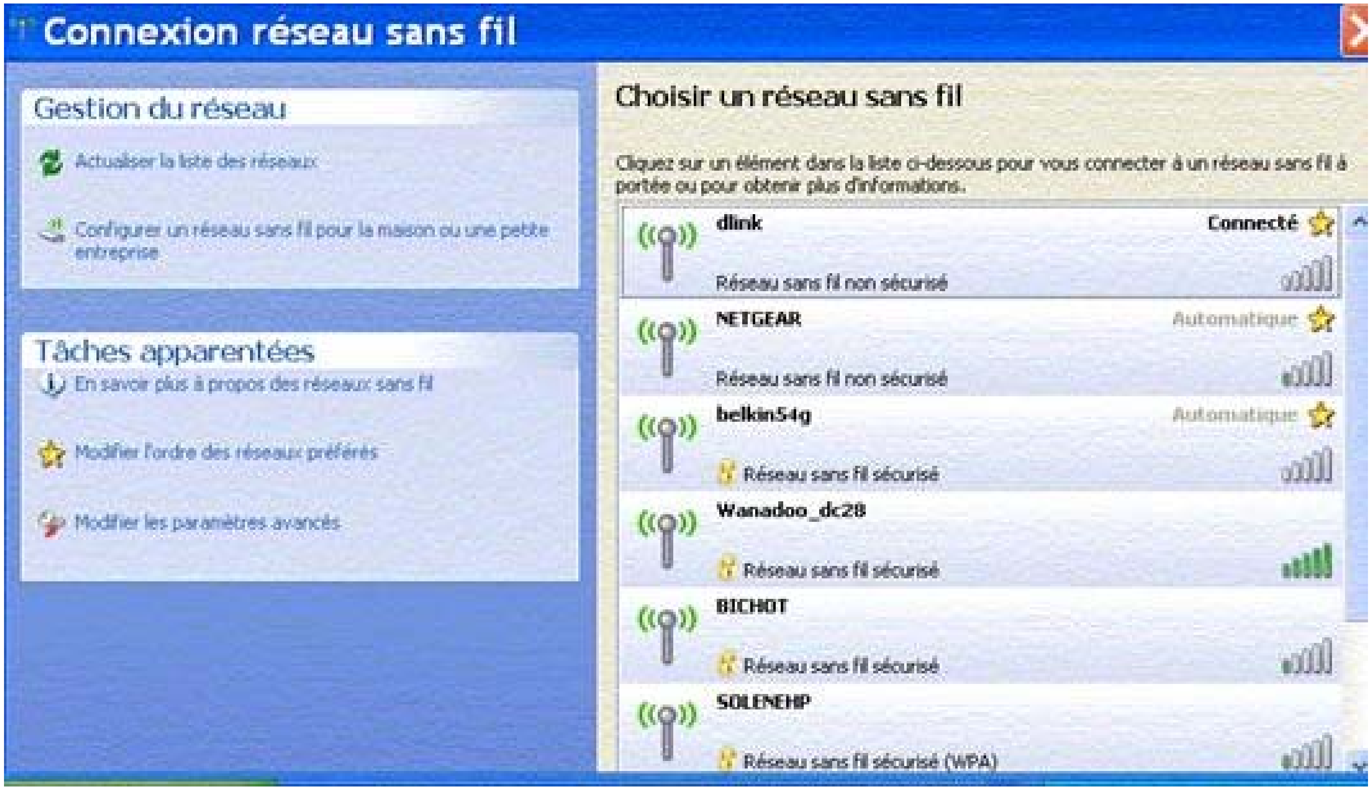



Figure 2: The association problem: the display

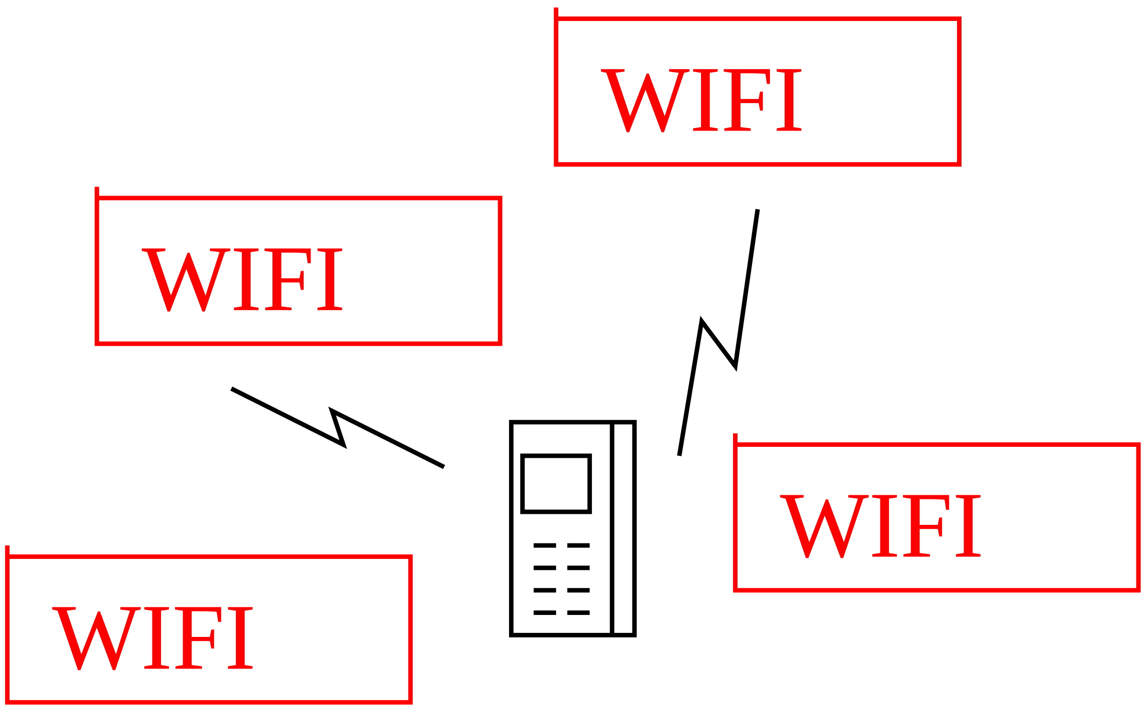



Figure 3: The Association Problem (2)

## The Association Problem

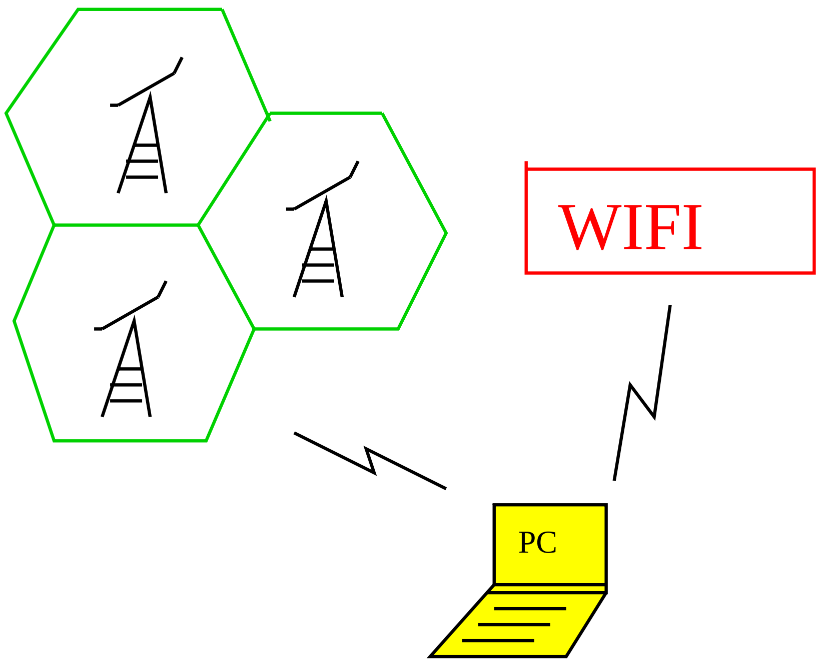
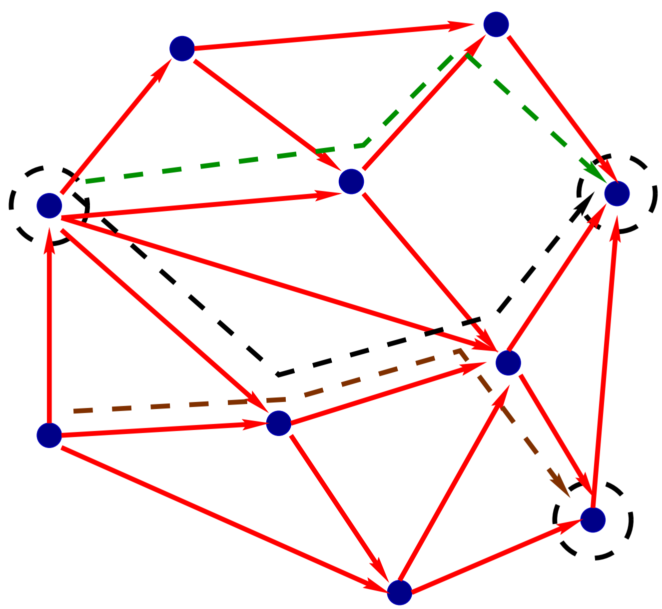



Figure 4: The Association Problem (3)

## 1.2 Routing

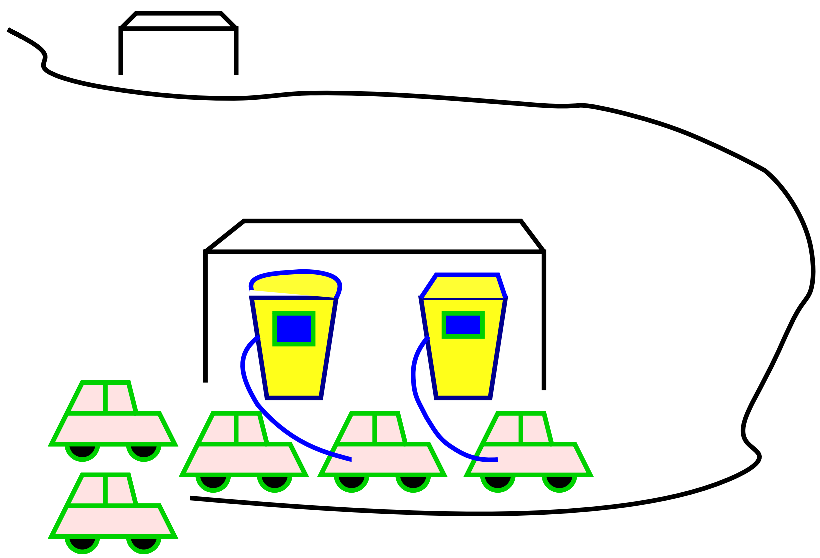


•Objectives:

- **Cooperative case:** minimize global cost.
- **non-cooperative case:** minimize individual cost.

## 1.3 The gaz station problem.

A car arrives at a gaz station and observes a line of waiting cars. Should the car wait as well or should it continue to the next az station?



•Application in networking: routing decision with partial information

## 1.4 Timing

Examples:

•When to arrive to the bank? A bank opens between 9h00 to 12h00. When should one come so as to minimize the expected waiting time?

•When to retry to make a phone call?

•**REF:** R. Hassin and M. Haviv, *To queue or not to queue: equilibrium behavior in queueing systems*, Kluwer Academic, 2002.

## 1.5 Classifying networking games in road traffic

**Individual choice of routes: road traffic**

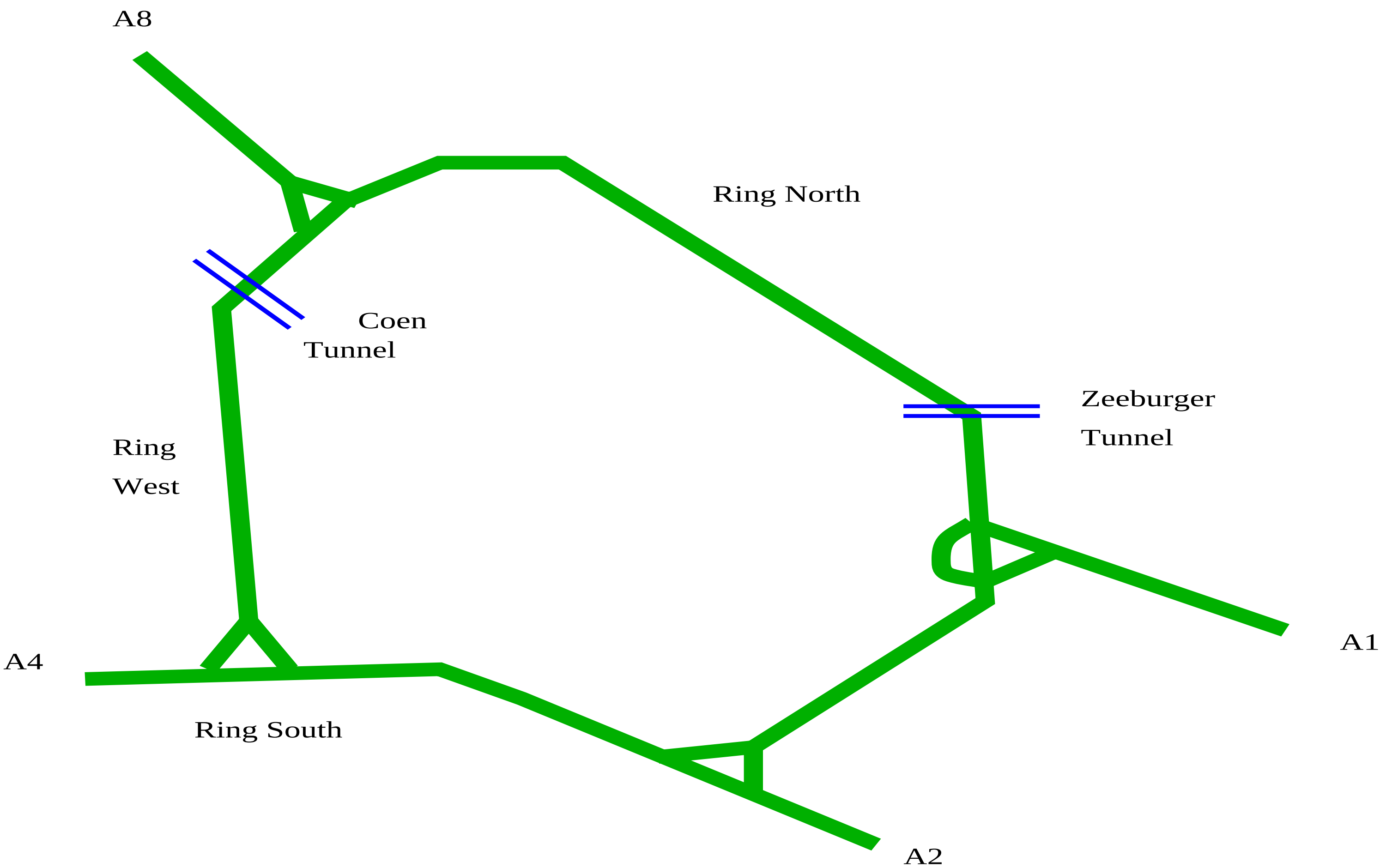
- A large number of vehicles choose their route every day
- Each vehicle has a source and a destination.
- Typical objective of a vehicle: minimize delay
- Optimality concept: Wardrop Equilibrium

## Example: The highway around Amsterdam

European project DRIVE II:

The vehicles have to decide the direction on the ring.

Objective: signalisation



## Grouped choice of routes

• Example: transportation companies

• The influence of the choice of routes for a large number of vehicles of a company has a non-negligible influence on delays of other vehicles.

• Optimality concept: Nash Equilibrium.

## 1.6 Classifying networking games in telecom

**Individual choice of routes**

•Ex: a large number of packets are transmitted over the network, and each packet makes its own routing choice.

•Ex: A large number of sessions share the network.

•A typical objective would be to minimise delays.

**Grouped choice of routes**

•Ex: A WEB application opens simultaneously several sessions (e.g. several images).

•Ex: Sessions cannot chooses their routes; the choice is done by the service provider for all its subscribers.

# 2 Introduction to non-cooperative games

## 2.1 Example: The Prisoner's Dilemma

- Emil and France are suspected of a crime.
- If both admit, they get 1 year in prison each.
- If both do not admit, they get 10 year in prison each.
- If one admits, he gets 15 years and the other is freed.

| EMIL \ FRANCE | 1 | 2 |
| --- | --- | --- |
| I | (1,1) | (15,0) |
| II | (0,15) | (10,10) |

Figure 5: Prisoner's Dilemma

•The solution (1,1) is unstable: each prisoner may try to improve by deviating. Only stable solution is (10,10).

•**Nash Equilibrium:** A "stable" solution: no player has an incentive to deviate unilaterally.

•**Pareto Equilibrium:** A cooperative "dominating solution": there is no way to improve the performance of one player without harming another one.

## 2.2 Inefficiency of Nash equilibria

| | | FRANCE | |
|---|---|---|---|
| | | 1 | 2 |
| EMIL | I | (1,1) | (15,0) |
| | II | (0,15) | (10,10) |

Pareto Equilibrium

Nash Equilibrium

- Nash Equilibria may be very inefficient!
- (10,10) is called an equilibrium in **pure strategies**

## 2.3 Non-monotonicity of Nash equilibria

Assume we change the "0" to "5": a prisoner that does not admit is always punished (we may have prior information that the crime was committed together).

**FRANCE**

| EMIL | | 1 | 2 |
|---|---|---|---|
| | I | (1,1) | (15,5) |
| | II | (5,15) | (10,10) |

(1,1) is Nash equilibrium and unique Pareto solution!

## 2.4 Pure and Mixed Nash equilibria

| | | Player II | |
|---|---|---|---|
| | | L | R |
| Pl. I | T | 0 (0) | 1 (−1) |
| | B | 1 (−1) | 0 (0) |

•Matrix U: the utility for player 1 is one if his action is different than that of player $j$. Else it is zero. Player 2 wants to minimize that utility: zero-sum game.

•**No pure Nash equilibrium!** Consider mixed actions: $p = (p_T, p_B)$ and $q = (q_L, q_R)$ are probabilities to choose actions by the players. The utility for player 1 is $\mathcal{U}(p, q) = pUq^T$.

•$p = q = (1/2, 1/2)$ is an equilibrium. With mixed policies, **every matrix game has an equilibrium**

## 2.5 Zero-sum games

•The utilities sum to zero for any $p$ and $q$.

•Define the **upper value**: $\overline{U} := \inf_q \sup_p U(p,q)$

•Define the **lower value**: $\underline{U} := \sup_p \inf_q U(p,q)$.

•We always have $\underline{U} \leq \overline{U}$ If equal then they are called the **value** of the game.

•For any pair $(p^*, q^*)$,

$$\inf_q U(p^*,q) \leq \sup_p \inf_q U(p,q) \leq \inf_q \sup_p U(p,q) \leq \sup_p U(p,q^*)$$

•A pair $(p^*, q^*)$ is a **saddle point** if $\inf_q U(p^*,q) = \sup_p U(p,q^*)$ and then all the inequalities become equalities. **Every 0-sum matrix game has a saddle point**.

# 3 Randomization and Correlated equilibria

| | action $1.a$ | action $1.b$ |
|---|---|---|
| action 2.i | $2, 1$ | $0, 0$ |
| action 2.ii | $0, 0$ | $1, 2$ |

•[Ex: Aumann] We now allow randomization. A player maximizes its **expected** value. Player 1 chooses columns, Player 2 chooses rows.

•Two pure Nash equilibria:
(1.a , 2.i) with the values (2,1), and (1.b , 2.ii) with values (1,2).

•The randomized strategy (1/3,2/3) for player 1 and (2/3,1/3) for player 2 is (the unique) symmetric equilibrium. The corresponding values are

$$(2/9) \times 1 + (2/9) \times 2 = 2/3 \quad \text{for each player}$$

## 3.1 Correlated equilibria: an example

| | action $1.a$ | action $1.b$ |
|---|---|---|
| action 2.i | $2, 1$ | $0, 0$ |
| action 2.ii | $0, 0$ | $1, 2$ |

Table 1: A matrix game [Aumann]

•Suppose that an arbitrator **suggests** to both either $(1.a,\ 2.i)$ (w.p. 1/2) or $(1.b,\ 2.ii)$ (w.p. 1/2 ).

•If the players follow the advise they obtain in expectation 3/2 each. Cannot be obtained without coordination.

## 3.2 Correlated games: discussion

•The game is still non-cooperative: no "binding contract".

•Here, the correlated equilibrium does not dominate the pure Nash equilibria.

•There are cases where it dominates all other Nash equilibria.

•The coordinator does not have to know anything about the game: all it has to do is flip a coin and send signals.

•Correlation is needed for **coordination**. Useful not just in games but also in team problems (a common objective).

•**Ref: R. J. Aumann,** *Journal of Mathematical Economics*, **1974.**

## 3.3 Applications: ALOHA (N. Bonneau, M. Debbah, E.A.)

•**Players:** $N$ mobile stations transmitting to a Base Station.

•If more than one mobile attempts transmission at the same time, the packets are lost.

•**Strategies:** transmission probabilities.

•**Objective:** maximize throughput, constraints on avg power.

•**Correlation mechanism:** Each mobile has a serial number within $1, ..., K$ where $K << N$. BS transmits a random number $k$ in $1, ..., K$.

•**Candidate equilibrium:** Mobiles with $k' \neq k$ will not attempt transmission. The others will transmit with some probability (to determine).

# 4 Constrained games

## 4.1 Routing: orthogonal constraints

•Network represented as a graph.

•A link has a **cost** (delay) per flow unit, that depends on the total flow through it.

•$K$ classes, each class $i$ has a fixed demand $\phi_i$ to ship from a source to a destination. A user splits its demand between paths according to its strategy.

•The cost of a path is the sum of the link costs.

•**constraints:** (i) non-negative flows and (ii) conservation constraints: for each node and class, the sum of entring class flow equals the sum of its leaving flows.

•These are **Orthogonal constraints:** The constraints of class $i$ do not depend on the strategies of other classes.

## 4.2 Routing: Common capacity constraints

•Model as before with additional constraints:
Each link $\ell$ has a **capacity constraint**: the total flow over it cannot exceed $C_\ell$.

•Link $\ell$ has a **Common Capacity Constraint** (CCC) for users in a group $M(\ell)$ if violation of the capacity of link $\ell$ for some $i \in M(\ell)$ implies its violation $\forall j \in M(\ell)$.

•$\ell$ has a CCC for some $M(\ell)$ iff for any multistrategy satisfying the link constraints, $\forall i \in M(\ell)$ send positive traffic on $\ell$.

•**Interpretation:** If the capacity of link $\ell$ is exceeded then all those using it suffer (delay or losses). Users not using the link are not affected.

•CCC are non-orthogonal. The constraints of class $i$ depend on the strategies of other classes.

**•2 Users Example (fig 1):**

•Each user has a direct dedicated path and a second path that traverses a common bus.

•Let $\phi_1 = \phi_2 = 4$, $C_l = 3, \forall l$. Then $\ell = (S1 - S2)$ has a CCC for $M(\ell) = \{1, 2\}$.

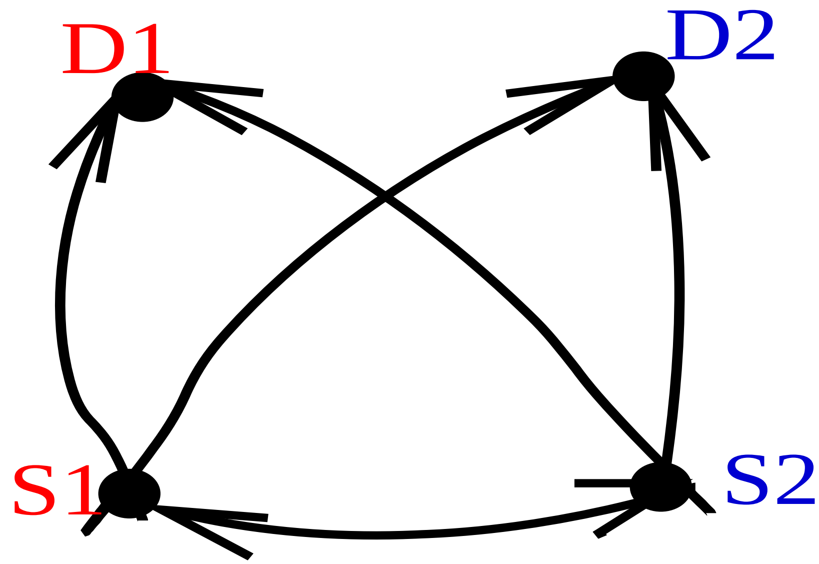
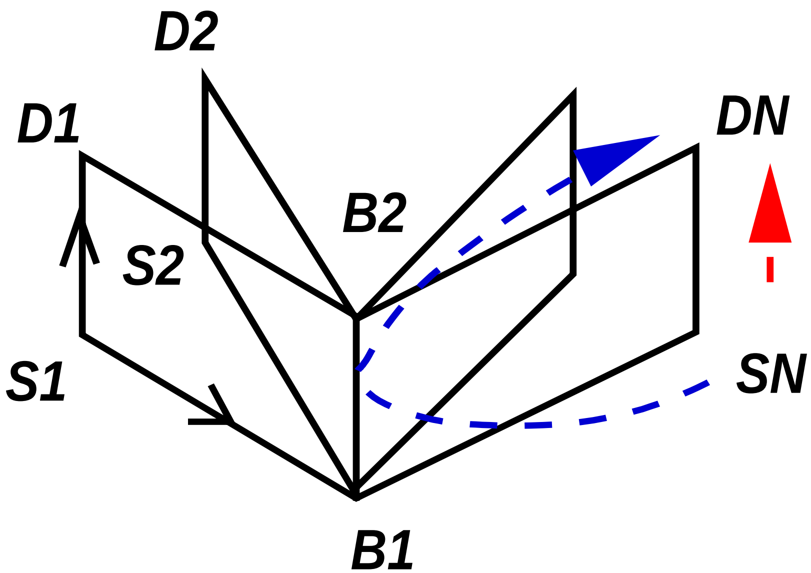


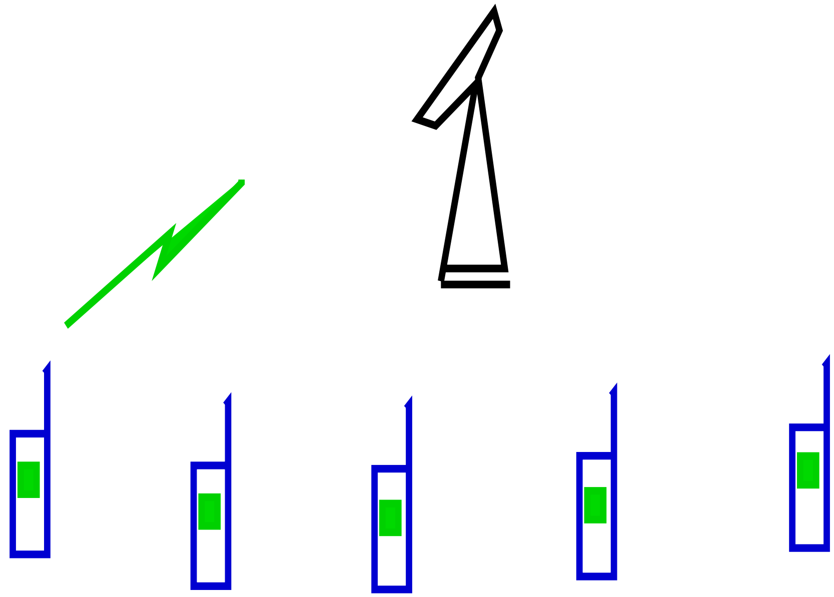

Figure 6: 2 users Example




•$N$ **users:** User $i$ has a **dedicated direct path** to the destination (link Si–Di) and an **alternative path** Si–Bi–B2–Di that shares a common bus B1–B2 with the others.
•**Example:** assume the demand $\phi^i$ of player $i$ is of $i$ units and that the capacity of each direct links Si–Di is $k$. As $k$ decreases, the number of users that have to ship flow over the common bus increases.
•Hence **the bus is a CCC** for all users in a group $M(S_k)$ where $S_k$ **decreases in** $k$. It is given by $S_k = \{N, N-1, ..., N-k\}$.

## 4.3 Power control: general constraints

$N$ mobile terminals. Each minimizes its transmission power $P_i$.



•The transmissions of all mobiles are received at a base station (BS).

• The received power of mobile $i$ is $h_i P_i$. $\xi$ is the power of the thermal noise at BS.

• The quality of the call of mobile $i$ is determined by the SINR (Signal to Initerference and Noise Ratio):

$$SINR_i(\mathbf{P}) = \frac{h_i P_i}{\sum_{j \neq i} h_j P_j + \xi}, \qquad \mathbf{P} = (P_1, ..., P_N).$$

• Each mobile $i$ seeks to **Minimize** $P_i$ subject to $SINR_i(\mathbf{P}) \geq \rho_i$.

• The consstraint of a user depends on strategies of other users.

• The contstrainsts are non-orthogonal and non common.

## 4.4 Constraint types

•Orthogonal Constraints

•Rosen's coupled constraints: all constraints are common to all players. A each constraint is feasible for a player iff it is for all players. **Refs:**

**J. Rosen**, Existence and uniqueness of equilibrium points for concave n-person games, 1965.

•General constraints. **Refs:**

**G. Debreu**, "A social equilibrium existence theorem", 1952

**G. Debreu**, "Existence of competitive equilibrium", 1982

# 5 Rosen constraints: non-existence of value

## 5.1 Example: matrix game

Player II

| Pl. I | L | R |
|---|---|---|
| T | 0 (2) | 1 (0) |
| B | −1 (0) | 0 (0) |

- Player 1 (2) chooses $T$ or $B$. ($L$ or $R$, respectively).
- The strategies of player 1 and 2 are the probabilities written as row vectors: $x = (x(T), x(B))$ and $y = (y(T), y(B))$, respectively.
- There are two matrices: $U$ - utilities and $D$ - constraints. The entries of $U$ (of $D$) are in black (in red, resp.).

- Player I maximizes the expected outcome $xUy'$ and player II minimzes it. ($y'$ is the transposed of $y$).
- As in Rosen, *the constraint is common to both players*:

$$xDy' \leq \rho. \tag{1}$$

where $\rho$ is some constant taken to be 0 in this Example.

•Consider $x = (1,0)$ (choose $T$ with probability 1). In order for the constraint to hold, player II has to play $y = (0,1)$ (choose R with probability 1). Hence

$$\max_x \min_y xUy' = U(T,R) = 1.$$

•Next assume Player II chooses $y = 1$. To meat the constraint, Player I has to play B with probability 1. Hence

$$\min_y \max_x xUy' = U(B,L) = -1.$$

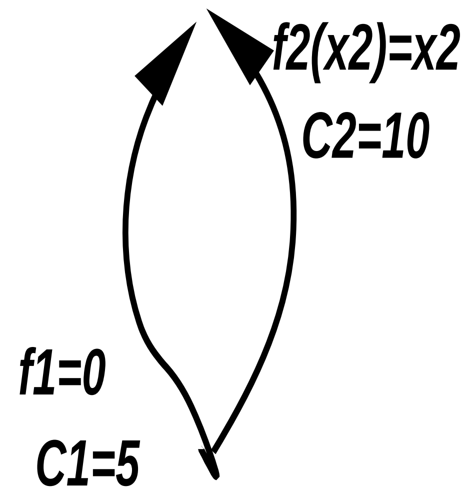
•We conclude that a value does not exist. Moreover, we obtain the surprising unusual inequality

$$\max_x \min_y xUy' > \min_y \max_x xUy'. \tag{2}$$

## 5.2 Networking game Example: parallel links

• $x_k^i$ is the amount of flow that player $i$ sends over link $k$.

• $f_k(x_k)$:= cost per unit flow of link $k$, where $x_k = x_k^1 + x_k^2$ is the total flow on link $k$.




•Let $f_1 = 0$, $f_2(x_l) = x_l$, $C_1 = 5, C_2 = 10$, $\phi^1 = \phi^2 = 3$.

•Player 1 maximizes $J(\mathbf{x}) = \mathbf{x_2^1}(\mathbf{x_2^1} + \mathbf{x_2^2})$ and player 2 minimizes it.

•If player 1 plays first then its dominating strategy - ship all his demand to link 1: $x_1^1 = 3, x_2^1 = 0$ and player, $J(\mathbf{x}) = 0$, so that $\min_{x^1} \sup_{x^2} J(\mathbf{x}) = 0$.

•The same holds for player 2, who has a dominating strategy of sending all her flow to link 1. Hence

$$\sup_{x^2} \min_{x^1} J(\mathbf{x}) = 1.$$

We thus get again the surprising inequality: $\min_{x^1} \sup_{x^2} J(\mathbf{x}) < \sup_{x^2} \min_{x^1} J(\mathbf{x})$..

# 6 Properties of zero-sum games with Rosen's constraints

•Consider a zero-sum game with a set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ of multi-strategies.

•$\mathcal{S}_i(z)$:= non-empty set of strategies available to user $i$ when the other player $j \neq i$ plays $z$.

•A multi-strategy $(x, y)$ is feasible if $x \in \mathcal{S}(y)$ and $y \in \mathcal{S}(x)$.

•Let the **unrestricted game** be the game with no consnstraints

**Theorem 6.1** *Assume that in the unrestricted game a value exists, i.e.*

$$\sup_{x \in \mathcal{S}_1} \inf_{y \in \mathcal{S}_2} U(x, y) = \inf_{y \in \mathcal{S}_2} \sup_{x \in \mathcal{S}_1} U(x, y) \tag{3}$$

*Then the original constrained game satisfies*

$$\sup_{x \in \mathcal{S}_1} \inf_{y \in \mathcal{S}_2(x)} U(x, y) \geq \inf_{y \in \mathcal{S}_2} \sup_{x \in \mathcal{S}_1(y)} U(x, y) \tag{4}$$

## 6.1 Remarks

•In constrained zero-sum games, the maximization and minimization by players 1 and 2 are restricted to **feasible multi-strategies**.
We can relax this: in the left hand side or (4), the maximization of player 1 is over all $S_1$; player 2 takes care that the constraints of player 1 are saatisfied.
A symmetric argument holds for the right hand side.

•(4) shows that the behavior observed for Rosen's constraints, where the "upper-value" is smaller than the "lower-value", is typical for constrained games of Rosen's type.

•(4) holds also in constrained games without Rosen's structure of common constraints. But it does not have any more a useful interpretation since

1. the "lower-value" (in the left hand side) is not restricted any more to multi-strategies that are feasible for player 1, and
2. the "upper-value" (in the right hand side) is not restricted any more to multi-strategies that are feasible for player 2.

# 7 Zero-sum games with general constraints

•We now raise the question of whether a value exists when the constraints do not satisfy the frameworok of Rosen

•We begin considerinig the case where only player 1 has constraints. These depend on the stgrategies of both players.

## 7.1 Formulation of the problem

Let $S_1$ be the set of strategies available to player 1. Player 2's strategies when player 1 uses strategy $x$ are given by the set $S_2(x) = \{y : D(x, y) \leq \rho\}$.

If player 2 knows $x$ then it can play any strategy in $S_2(x)$. If it does not know $x$ it is obliged to play a strategy $y$ that will guarantee that $\sup_{x \in \mathcal{S}_1} D(x, y) \leq \rho$.

Let $G_2 = \{y : \forall x,\ D(x, y) \leq \rho\}$. The general problem we consider is of determining the relation between the left and the right hand side of the following:

$$\sup_{x \in S_1} \inf_{y \in S_2(x)} U(x,y) \begin{array}{c} ? \\ > \\ = \\ < \end{array} \inf_{y \in G_2} \sup_{x \in S_1} U(x,y)$$

The following holds for any $x^* \in \mathcal{S}_1$, and $y^* \in G_2$:

$$\inf_{y \in S_2(x^*)} U(x^*,y) \le \sup_{x \in S_1} \inf_{y \in S_2(x)} U(x,y) \le \inf_{y \in G_2} \sup_{x \in S_1} U(x,y) \le \sup_{x \in S_1} U(x,y^*)$$

$(x^*, y^*)$ is said to be a value of the game where $x^* \in \mathcal{S}_1$, $y^* \in G_2$, if

$$\inf_{y \in S_2(x^*)} U(x^*,y) = \sup_{x \in S_1} U(x,y^*)$$

If it holds, then we conclude that

$$\inf_{y \in S_2(x^*)} U(x^*,y) = \sup_{x \in S_1} \inf_{y \in S_2(x)} U(x,y) = \inf_{y \in G_2} \sup_{x \in S_1} U(x,y) = \sup_{x \in S_1} U(x,y^*)$$

Note that the Rosen's joint coupled constraints are not special cases of this framework, so our previous Counterexamples do not allow us to conclude that a value does not exist here.

Symetric definitions hold when player 1 has constraints. Then if a value exists, i.e.

$$\inf_{y \in S_2} U(x^*, y) = \sup_{x \in \mathcal{S}_1(y^*)} U(x, y^*)$$

for some $x^* \in G_1$ and $y^* \in S_2$, then

$$\inf_{y \in S_2} U(x^*, y) = \sup_{x \in G_1} \inf_{y \in S_2} U(x, y) = \inf_{y \in S_2} \sup_{x \in S_1(y)} U(x, y) = \inf_{y \in \mathcal{S}_1(x^*)} U(x, y^*).$$

## 7.2 Matrix games

**Example 7.1** Zero-sum matrix game with constraints on player 1's strategies

•the constraint restricts only player 1 and not player 2.

•Set $\rho = 1/2$.

Player II

| Pl. I | L | R |
|---|---|---|
| T | 0 (1) | 1 (0) |
| B | 2 (0) | 0 (1) |

•$G_1$ is the singleton $(1/2, 1/2)$. Hence $\sup_{x \in G_1} \inf_{y \in S_2} U(x,y) = 1/2$, obtained when player 2 uses action $R$ w.p.1.

•Thus Player 1 can guarantee to receive a payoff of at least 1/2 if he plays first.

•If Player 2 plays first, he can guarantee that the payoff of Player 1 would not exceeed 2/3.

•We conclude that the following inequality holds:

$$\sup_{x \in G_1} \inf_{y \in S_2} U(x,y) < \inf_{y \in S_2} \sup_{x \in S_1(y)} U(x,y).$$

$\diamond$

•The inequality is the opposit than the one we had in Rosen's setting

# 2nd Part: Non-Atomic games

# with

# networking applications

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Winter School, March 2007

# 8 Non-atomic games

## 8.1 Network exampe

Network: a graph $G = (V, \mathcal{C})$ where

•$V$ is a set of nodes

•$L$ a class of directed links

$K$ classes of traffic.

•Class $i$ has a set $\mathbf{P_i}$ of paths, and traffic demand of $\phi^{(i)}$.

•Link cost: $f_l(y_l)$, strictly monotone increasing in the link flow $y_l$

## Definitions

•Strategies: the amount $x_p^{(i)}$ of traffic of class $i$ is sent over path $p$.

•Link traffic: $y_l = \sum_{p,i} \delta_{lp} x_p^{(i)}$ where $\delta_{lp} = 1$ if $l$ is on the path $p$.

•Flow constraints: for each class $i$, $\sum_{p \in \mathbf{P}(i)} x_p^{(i)} = \boldsymbol{\phi}^{(i)}$.

•Let $\gamma_p^{i,i'} = 1$ if $p \in \mathbf{P}(i')$ and $i = i'$, and 0 otherwize.

•Matrix form of flow constraints $\mathbf{\Gamma^T x} = \boldsymbol{\phi}$.

$\Gamma^T$ has $K$ rows and $\pi$ columns, where $\pi = \sum_i |P(i)|$.

$x$ is a column vector (size $\pi$). The first $P(1)$ elements correspond to the flow over the pathrs of class 1, etc.

## 8.2 Global optimization

•**Objective:**

$$\min_x \Delta(x) \quad \text{where } \Delta(x) := \frac{1}{D}\sum_l y_l f_l(y_l) \quad D{:=} \text{ total demand.} \tag{5}$$

s.t. (i) flow conservation, (ii) non-negative flows, (iii) $y$ in terms of $x$.

•Define $t_p^{(k)} = \partial(\Phi\Delta)/\partial x_p^{(k)}$, i.e., class $k$ marginal cost of $p$, $p \in \Pi^{(k)}$.

•$\mathbf{t} = [t_1^{(1)}, t_2^{(1)}, \ldots, t_1^{(2)}, t_2^{(2)}, \ldots]^{\mathrm{T}}$ is the gradient vector of the function $\Phi\Delta$.

•**Characterization of optimal solution through complementarity:**

$\mathbf{x}$ optimal iff there exist $K$ Lagrange multipliers $\boldsymbol{\alpha}$ such that

$$[\mathbf{t}(\mathbf{x}) - \boldsymbol{\Gamma\alpha}] \cdot \mathbf{x} = 0, \tag{6}$$

$$\mathbf{t}(\mathbf{x}) - \boldsymbol{\Gamma\alpha} \geq 0, \tag{7}$$

$$\boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{x} - \boldsymbol{\phi} = 0, \tag{8}$$

$$\mathbf{x} \geq 0. \tag{9}$$

**•Alternative characterisation: Variational inequalities**

$\bar{\mathbf{x}}$ is an optimal solution iff

$$\begin{aligned} \mathbf{t}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}}) &\geq 0, \quad \text{for all } \mathbf{x} \\ \text{such that } \boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{x} &= \boldsymbol{\phi} \text{ and } \mathbf{x} \geq 0. \end{aligned} \tag{10}$$

## 8.3 Wardrop equilibrium

•Non-atomic setting: large number of non-cooperative players.

•Each player has a negligible influence on others performances.

•$\mathbf{x}$ is a Wardrop equilibrium if each player uses a least costly.
$T_p^{(k)}(\mathbf{x}) :=$ cost of path $p$ for class $k$ user. Equals sum of link costs along $p$.

•A type $k$ user chooses a path $\hat{p}$ that satisfies

$$T_{\hat{p}}^{(k)}(\mathbf{x}) = \min_{p \in \Pi^{(k)}} T_p^{(k)}(\mathbf{x}) =: A^{(k)} \tag{11}$$

Thus $\mathbf{x}$ is a Wardrop equilibrium if

$$T_p^{(k)}(\mathbf{x}) \geq A^{(k)}, \quad x_p^{(k)} = 0, \tag{12}$$

$$T_p^{(k)}(\mathbf{x}) = A^{(k)}, \quad x_p^{(k)} > 0, \tag{13}$$

and the flow constraints hold (conservation, nonnegativity). Matrix notation:

$$[\mathbf{T}(\mathbf{x}) - \mathbf{\Gamma A}] \cdot \mathbf{x} = 0, \quad (14)$$

$$\mathbf{T}(\mathbf{x}) - \mathbf{\Gamma A} \geq 0, \quad (15)$$

$$\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x} - \boldsymbol{\phi} = 0, \quad (16)$$

$$\mathbf{x} \geq 0, \quad (17)$$

where $\mathbf{A} = [A^{(1)}, A^{(2)}, \ldots, A^{(K)}]^{\mathrm{T}}$,

•Define the **potential** $G(\mathbf{x}) = \frac{1}{D}\sum_l \int_0^{y_l} f_l(s)ds$. Then

$$T_p^{(k)}(\mathbf{x}) = \frac{\partial}{\partial x_p^{(k)}}(DG(\mathbf{x})).$$

We get the same conditions for optimality of $\mathbf{x}$ in the global optimization problem where link costs are replaced by their integral.

•**Conclusion:** $\mathbf{x}$ is a Wardrop equilibrium if it solves a **global optimization** problem:

$$\text{minimize } G(\mathbf{x}) \qquad \text{s.t. (16)-(17).}$$

**•Alternative characterisation: Variational inequalities**

$\bar{\mathbf{x}}$ is a Wardrop equilibrium if and only if it is feasible and

$$\begin{aligned} \mathbf{T}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}}) &\geq 0, \quad \text{for all } \mathbf{x} \\ \text{such that } \mathbf{\Gamma}^{\mathrm{T}} \mathbf{x} &= \boldsymbol{\phi} \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

•Again the same form of the global optimization provided that $\mathbf{T}(\bar{\mathbf{x}})$ is interpreted as the gradient of a potential.

## 8.4 Applications to adhoc networks

•P. Gupta and P. R. Kumar propose a new routing algorithm for Adhoc networks in IEEE CDC, 1997

•No real players, no game.

•This is a shortest path (delay) protocole: each packet that arrives at a mobile is forwarded so as to minimize its delay.

•Reason: to decrease number of out-or-order packets and to minimize resequencing delay

## 8.5 Limitations of the model (i)

•**Link correlations:** The cost over link $\ell$ may depend on the flow over other links,

•**Road traffic Examples:** (i) Two way traffic. Congestion in one direction could contribute to congestion in the other direction [Dafermos, Transportation Sc. 1971].

•**Data network traffic:** A congestion of TCP connections in one direction impacts on the flow of ACKs in the opposite direction.

•**Wireless context:** Links are radio channels. They can have mutual interference.

## Limitations of the model (ii): multiclass traffic

•(i) Link cost may **depend on the flow of each user** $\{y_l^{(i)}\}$ rather than on the total link flow.

•(ii) Moreover, the link cost may differ from one class to another.

•**Road traffic example:** bicycles, cars and trucks contribute differently to congestion, and may experience congestion differently. [Dafermos, The traffic assignment problem for multiclass-user transportation networks. Transp Sci 1972]

•**Data networks example:** Packets of different size contribute differently to congestion.

•**Example:** Diffserv provides priority to some traffic over other. The priority traffic encounters less congestions, but is more expensive.

## Potential game with continuouos players set

•The networking game is said to be a potential game if there is a continuously differentiable function $G : \mathcal{X} \to R$ such that for all $i$ and $\ell$,

$$\frac{\partial G(\mathbf{x})}{\partial x_\ell^i} = f_\ell^i(\mathbf{x})$$

•the Wardrop equilibrium is obtained by maximizing the potential

## Conditions for existence of a Potential

•Assume that the matrix

$$\left[\frac{\partial f_l^j(\mathbf{x})}{\partial x_k^i}\right]$$

is **symmetric** and **positive definite**. $f_l^j$ is the cost of link $l$ for class $j$.

•A potential $G(\mathbf{x})$ exists, it is the sum (over the entries corresponding to links and users) of the line integral

$$G(\mathbf{x}) = \sum_{l,j} \int_0^{\mathbf{x}} f_l^j(\mathbf{s}) d(\mathbf{s})$$

which is path independent.

•There is a "unique" Wardrop equilibrium (if the link costs are monotone).

•Major research problem: what to do when there is no potential.

## Limitations of the model (iii): Non-additive costs.

•Example: **loss networks** work with El-Aazouzi and Abramov.

•$\mathcal{C}$ resources. Resource $c$ has $R_c$ capacity units (integer).

•There are $N$ classes of calls ($\mathcal{N} = \{1, 2, .., N\}$),

•Associated with class $n$ are

- Arrival rate $\lambda_n$, and an average holding time $\mu_n^{-1}$,
- Bandwidth requirement, $b_n$ integer units.
- Route $r_n \subseteq \mathcal{C}$.

Denote $\rho_n = \lambda_n / \mu_n$ the class $n$ workload.

## Blocking probabilities

- Let $\mathcal{N}_c$ the subset of classes that use resource $c$,

$$\mathcal{N}_c = \{n \in \mathcal{N} : c \in r_n\}. \tag{18}$$

- Let $m_n$ the number of calls of class $n$ in the system, and

$$\mathbf{m} = (m_1, m_2, ..., m_N).$$

- The state space is $\mathcal{X} = \Big\{\mathbf{m} : \sum_{n \in \mathcal{N}_c} b_n m_n \le R_c, \quad c \in \mathcal{C}\Big\}$.
- Let $\mathcal{X}_n$ the subset of states for which there is an available bandwidth for another arrival of a class-$n$ call:

$$\mathcal{X}_n = \Big\{\mathbf{m} \in \mathcal{X} : \sum_{i \in \mathcal{N}_c} b_i m_i \le R_c - b_n, c \in r_n\Big\}. \tag{19}$$

The steady state distribution is

$$\mathbf{P}\{\mathbf{X}=\mathbf{m}\} = \frac{1}{G}\prod_{n=1}^{N}\frac{\rho_n^{m_n}}{m_n!}, \qquad \mathbf{m}\in\mathcal{X}, \tag{20}$$

where

$$G = \sum_{\mathbf{m}\in\mathcal{X}}\prod_{n=1}^{N}\frac{\rho_n^{m_n}}{m_n!}. \tag{21}$$

The probability of blocking of a class-$n$ call is

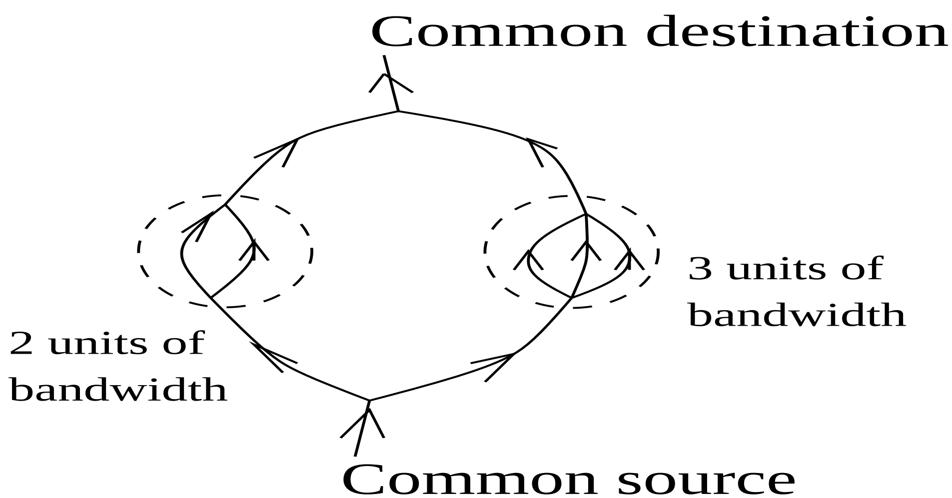
$$B_n = 1 - \frac{\sum_{\mathbf{m}\in\mathcal{X}_n}\prod_{i=1}^{N}\rho_i^{m_i}/m_i!}{\sum_{\mathbf{m}\in\mathcal{X}}\prod_{i=1}^{N}\rho_i^{m_i}/m_i!}. \tag{22}$$

Group of users

- Consider $L$ groups which split their demands via the networks.
- Group $l$ can use any one of subset $\mathcal{N}^l \subset \mathcal{N}$ of classes. The set $\mathcal{N}^l$ is characterized by a common source and destination as well as a common parameters $b_l$, $\mu_l$ and $\lambda_l$.
- **Strategies:** Group $l$ sends a fraction $p_{l,n}$ of its demand via the route $r_n$.

## Non-uniqueness of Wardrop equilibrium

Consider the following example:




- There are two parallel links :
  - The first one, $a$, has a capacity of 2 bandwidth units
  - The second one , $b$, has a capacity of 3 bandwidth units

- There are 2 groups:
  - The calls of group I require 1 bandwidth units.
    $\mathcal{N}^1 = \{(I,a),(I,b)\}$, $b_{(I,a)} = b_{(I,b)} = 1$.
  - The calls of group II require 2 bandwidth units.

    $\mathcal{N}^2 = \{(II,a),(II,b)\}$, $b_{(II,a)} = b_{(II,b)} = 2$.
- Two groups can send traffic through both links.
  We have then 4 classes with the same source and destination :

$$\mathcal{N} = \{(I,a),(I,b),(II,a),(II,b)\},$$

where $l = I, II$ is the group and $j = a, b$ is the link.

## Results

- We obtained three different Wardrop equilibria.
- In general, no potential

## Parallel links with equal bandwidth requirements

The blocking probability over link $i$, is given by the Erlang loss formula:

$$B_i(\lambda(i)) = \frac{\lambda(i)^{R_i}/R_i!}{\sum_{j=0}^{R_i} \lambda(i)^j/j!}. \qquad \text{(Erlang B formula)}$$

Let $\Lambda = \sum_{l=1}^{L} \lambda_l$. The game has a potential:

$$G(\lambda) := \sum_{i \in \mathcal{N}} \int_0^{\lambda(i)} B_i(z)dz = -\sum_{i \in \mathcal{N}} \log g_{R_i}(\lambda(i)), \quad \text{where} \quad g_r(x) = \sum_{i=0}^{r} x^i/i! \tag{23}$$

where $\lambda = (\lambda(i), i \in \mathcal{N})$, by solving :

$$\min G(\lambda) \quad s.t. \sum_{i \in \mathcal{N}} \lambda(i) = \Lambda,\ \lambda(i) \geq 0,\ \forall i = 1, ..., N. \tag{24}$$

• Unique Wardrop equilibrium

• We can interpret the Wardrop equilibrium as the proportional fair assignment.

## 8.6 Convergence [A. Kumar, S. Shakkottai, E.A.]

•For a strategy $\mathbf{y}_n$ of class $n$, Define $\mathcal{S}_n(\mathbf{y})$ all the pure strategies in its support.

•**Definition [Sandholm]**: The dynamic

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{V}(\mathbf{y})$$

is said to be PC (Positively Correlated) if

$$\sum_{n \in \mathcal{N}} \sum_{p \in \mathcal{S}_n(\mathbf{y})} T_p^i(\mathbf{y}) V_p^i > 0 \quad \text{whenever } V(\mathbf{y}) \neq 0.$$

•If $V$ satisfies PC then all Wardrop equilibria are stationary points.

•**Replicator dynamics.**

$$\frac{\mathrm{d}y_p^n}{\mathrm{d}t} = \mathbf{V}(\mathbf{y}) = y_p^n\Big(T_p^n(\mathbf{y}) - \frac{1}{\boldsymbol{\phi}^{(n)}}\sum_{q\in\mathcal{S}_n(\mathbf{y})} y_q^n T_q^n(\mathbf{y})\Big)$$

For each $n$, the dynamics is between utilization of resources (paths) of class $n$.

**Results:**

•the replicator dynamics satisifies PC

•PC is also satisfied by the dynamics called "Brown - von Neumann - Nash"

•PC is satisfied by mixtures of the above (each class may follow either one of the schemes.

## 8.7 Applications: the association problem

•We have developed simple formulaes for the asymptotic capacity of IEEE802.11

•We have developed simple formulaes for the asymptotic capacity of CDMA

•The association problem can now be defined as finding the equilibrium in problem of $C$ parallel links.

•A link represents either WLAN or cellular CDMA.

•Mobiles of given class can choose which link to use among those available for that class.

•This is a multi-class potential game.

# 9 Braess paradox

Adding a link, or adding capacity to a link, increases delay for all users

Assume: route 1-3, as well as 2-4 are used.

$$g(x_1) + f(x_3) = f(x_2) + g(x_4). \qquad (25)$$

We now express $x_5$:

$$x_1 - x_3 = -x_2 + x_4.$$

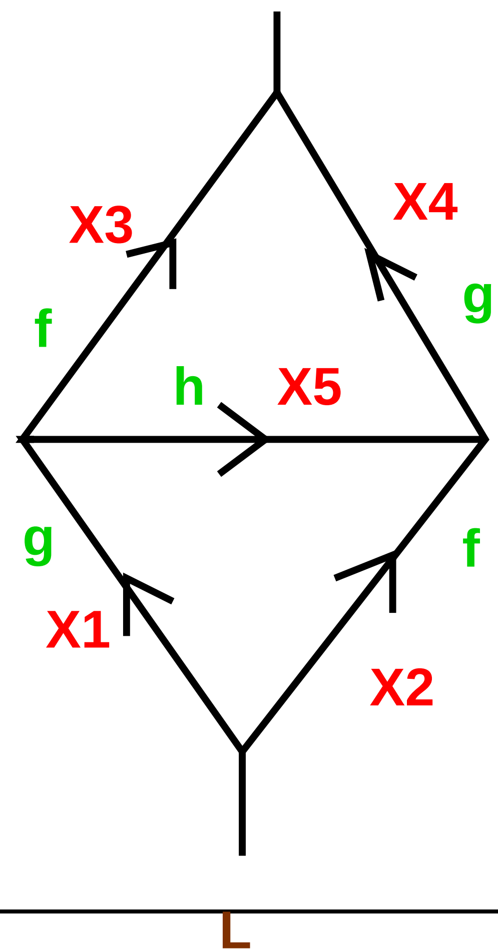
If $f$ is linear then this implies

$$f(x_1) - f(x_3) = -f(x_2) + f(x_4).$$

Summing with (25), we get

$$f(x_1) + g(x_1) = f(x_4) + g(x_4).$$

If $f+g$ is strictly increasing then $x_1 = x_4$. Hence also $x_2 = x_3$.



Now, $x_2 = L - x_1$. Hence

$$x_5 = x_1 - x_3 = x_1 - x_2 = 2x_1 - L.$$

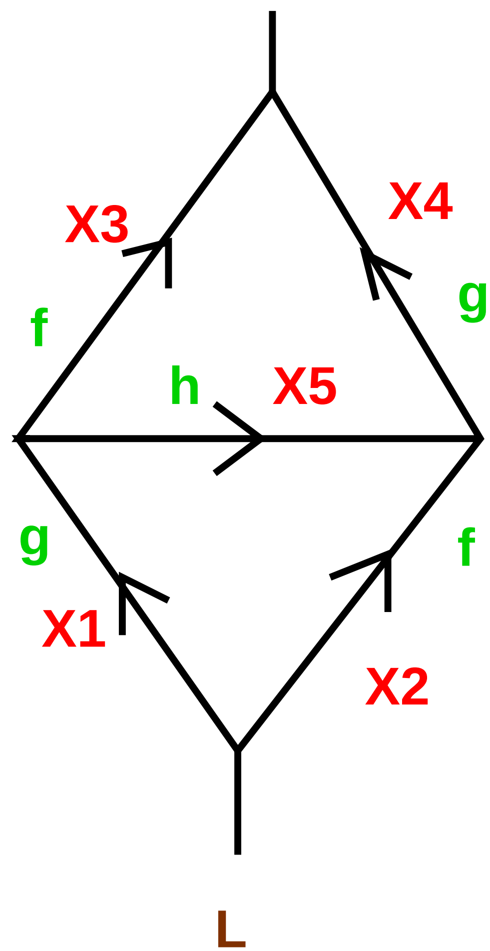
If route 1-5-4 is also used then

$$g(x_1) + h(x_5) = f(x_2).$$

We conclude that

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

This gives $x_1$.

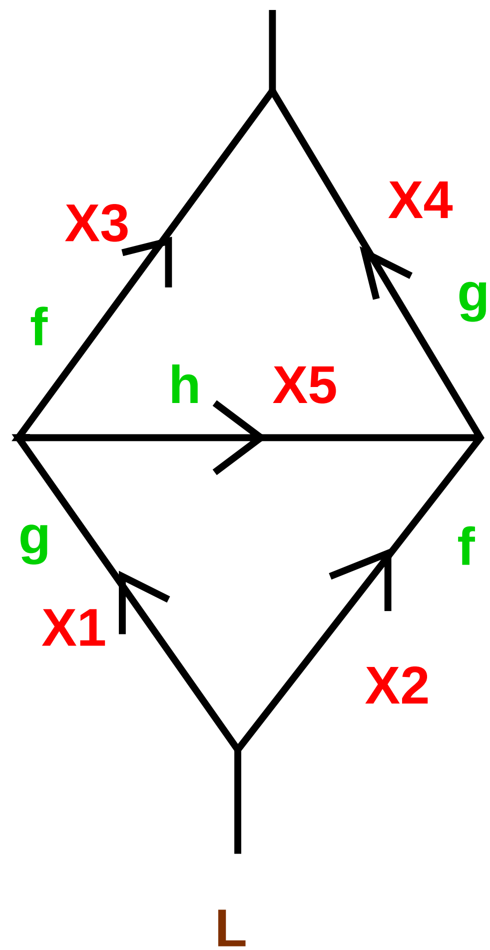


# Original Example of Braess

•Choose $L = 6$, $f(x) = 50 + x$, $g(x) = 10x$, $h(x) = \infty$.
Then $x_1 = x_2 = 3$,

$$D_{13} = D_{24} = 83.$$

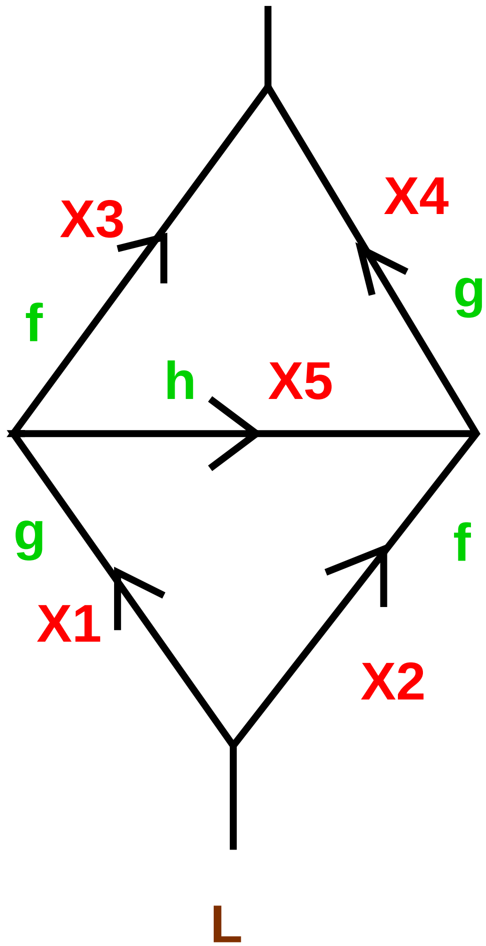
•Take $h = 10 + x$.
With $x_1 = x_3 = 3$, $D_{154} = 70 < D_{13}$. Not Wardrop Equilibrium!



•Suppose 1 unit moves from 2-4 to 1-5-4.
Then

$D_{1-3} = 40{+}53 = 93, \quad D_{2-4} = 52{+}30 = 82,$

$$D_{1-5-4} = 40 + 11 + 40 = 91.$$

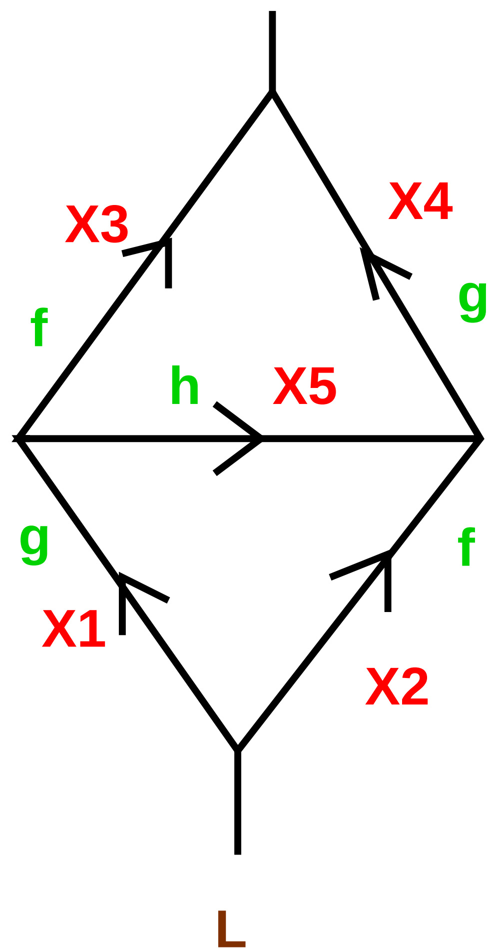


•Suppose 1 unit moves from 1-3 to 1-5-4.
We have $x_1 = 4, x_2 = 2, x_5 = 2.$
Then

$$D_{1-3} = 40{+}52 = 92, \quad D_{2-4} = 52{+}40 = 92,$$

$$D_{1-5-4} = 40 + 12 + 40 = 92.$$

This satisfies Wardrop conditions!



We now check the equation:

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

We get:

$$10x_1 + (10 + 2x_1 - 6) = (50 + 6 - x_1)$$

Hence $x_1 = 4$, $x_2 = L - x_1 = 2$, $x_5 = x_1 - x_2 = 2$.

# 10 Potential games: the discrete player set

•A function $G : \mathcal{S} \rightarrow R$ is called **a potential** if for every user $i$ and every strategy $\mathcal{S}$, and for every strategy $v_i$ for user $i$

$$G([s_i, S_{-i}]) - G([v_i, S_{-i}]) = U_i([s_i, S_{-i}]) - U_i([v_i, S_{-i}]). \tag{26}$$

•Assume that $U_i$ are continuously differentiable. An equivalent condition for $G$ to be a potential is:

$$\frac{\partial G(\mathbf{s})}{\partial s_i} = \frac{\partial U_i(\mathbf{s})}{\partial s_i} \qquad \forall i \in \mathcal{N}$$

## 10.1 Discrete congestion games [Rosenthal 1973]

•Set $\mathcal{C}$ of **resources** (e.g. links), or of **facilities**. $N$ users.

•**Strategy set** $\Sigma^i$ **of user** $i$**:** Each $s_i \in \Sigma^i$ is a subset of resources.

•Define the payoff $d_j(k)$ for each user of resource $j$ given that $k$ users share the resource. Further define by $k^j(S)$ the set of users of resource $j$ under strategy $S$. Define the **payoffs:**

$$v^i(S) = \sum_{j \in s_i} d_j \Big( |k^j(S)| \Big).$$

•**Every congestion game is a potential game.** [Monderer and Shapley, 1996].

## 10.2 Convergence.

•Introduce the **Single User Improving Policy** (SUIP) as follows. Consider any strictly increasing time sequence $T_n$ where at each $T_n$ one user changes its strategy so as to strictly improve its own utility.

•Under convexity conditions, every potential games has a **unique equilibrium** in pure strategies, to which the system converges in **finite time** from any initial point if an SUIP strategy is used by all players.

# 11 Concave games with constraints

•In absence of side-constraints, policies that a user can choose do not depend on the choice of other users. $U$ is **rectangular**.

•In networking games there can be CAPACITY constraints.
**Explicit constraint:** $\lambda_m \leq C_m$.
**Implicit:** e.g. **queueing delay cost**:

$$f_m(\lambda_m) = \frac{1}{C_m - \lambda_m} \text{ if } \lambda_m < C_m, \text{ otherwize infinity}$$

•If a capacity constraint is violated for one user, it is also violated for all user sharing the same link.

•The set of all multi-strategies need not be rectangular.
Instead, a **convex** multi-strategy set.

## 11.1 Concave games [Rosen 65]

•Assume that the utility $V_i$ of each player $i$ is continuous in its argument and **concave** in the strategy of player $i$.

•Assume that the set of strategies is convex and closed.

•Then an equilibrium exists

## 11.2 Uniqueness, orthogonal case [Rosen 65]

•Standard condition for concavity of a function $V$: denote

$$H_{ij} := \frac{\partial V^2}{\partial u_i \partial u_j}$$

If $H + H^T$ is strictly negative definite then $V$ is strictly concave, and has a unique minimum.

•The game case: let $V_i$ be the utility of player $i$. Define

$$H_{ij} := \frac{\partial V_i^2}{\partial u_i \partial u_j}$$

If $H + H^T$ is strictly negative definite then $V$ is STRICTLY DIAGONAL CONCAVE, there exists a unique equilibrium.

•There are alternative conditions using the monotonicity of the first order derivatives.

## 11.3 Constrained equilibria [Rosen 65]

•**Rosen's constrained equilibrium $u$:**
$u \in U$ is an equilibrium if no user can profit from unilaterally deviating **within** $U$.

•Constrained equilibria are in general non-unique.

•Pricing: write the constraints in a vector form as $g(u) \leq 0$.

•At equilibrium $u^*$, the policy $u_i$ of player $i$ maximizes its utility $V_i$. Equivalently, there exists a Lagrange multiplier $\gamma^i$ such that

$$u_i^* \in argmax\Big[J_i(u_i,(u^*)^{-i})\Big]$$

where $J_i = V_i + \sum_l \gamma_l^i g_l$.

•Complementarity property: $\gamma_l^i$ can be chosen zero except for $l$ for which the constraints are reached with equality.

•Pricing interpretation: we can replace the constraint by an additional price, and then, $u_i^*$ is obtained by minizing a non-constrained problem.

•Does not scale well since the price is a FUNCTION OF $i$.

•Example: networking games. We want a price that depends on the total link flow.

## 11.4 Normalized equilibria

•Assume that for any consants $a_i$, $\gamma_l^i$ has the form $a_i \gamma_l$. Then there is no scaling problem (e.g. $a_i = 1$).

•This is Rosen's "Normalized equilibrium".

•Rosen shows under SDC:
(i) In constrained concave games, there is a unique Normalized equilibrium for each $a$.
(ii) Convergence to the equilibrium.

# 12 S-modular games

•D. Topkis, SIAM J. Contr. Optim., 1979,

•David D. Yao, Queueing Systems, 1995.

•Assume that the strategy space $S_i$ of player $i$ is a compact sublattice of $R$.

•Definition: The utility $f_i$ for player $i$ is supermodular iff

$$f_i(x \wedge y) + f_i(x \vee y) \geq f_i(x) + f_i(y).$$

•If $f_i$ is twice differentiable then supermodularity is equivalent to

$$\frac{\partial^2 f_i(x)}{\partial x_1 \partial x_2} \geq 0.$$

## 12.1 Monotonicity of maximizers

•Let $f$ be a supermodular function. Then the maximizer with respect to $x_i$ is increasing in $x_j, j \neq i$.

•More preciesly, define

$$x_1^*(x_2) = argmax_{x_1} f(x_1, x_2).$$

Then $x_2 \leq x_2'$ implies $x_1^*(x_2) \leq x_1^*(x_2')$.

## 12.2 Monotonicity of the policy sets

•Consider 2 players. We allow $S_i$ to depend on $x_j$

$$S_i = S_i(x_j), \quad i, j = 1, 2, \quad i \neq j.$$

•Monotonicity of sublattices $A \prec B$ if for any $a \in A$ and $b \in B$,

$$a \wedge b \in A \qquad \text{and} \qquad a \vee b \in B.$$

•Monotonicity of policy sets We assume

$$x_j \leq x'_j \Longrightarrow S_i(x_j) \prec S_i(x'_j).$$

This is called the Ascending Property. We define similarly the Descending Property.

•Lower semi continuity $x_1^k \to x_1^*$ and $x_2^* \in S_2(x_1^*)$ implies the existence of $\{x_2^k\}$ s.t. $x_2^k \in S_2(x_1^k)$ for each $k$, and $x_2^k \to x_2^*$.

## 12.3 Existence of Equilibria and Round Robin algorithms

Assume lower semi-continuity and compactness of the strategy sets.

•Supermodularity implies monotone convergence of the payoffs to an equilibrium. The monotonicity is in the same direction for all players. (We need the ascending property).

•Similarly with submodularity (for 2 players), but the monotonicity is in opposite directions. (We need the desceding property).

•In both cases, there need not be a unique equilibrium.

•Extensions to costs that are submodular in some components and supermodular in others. Extensions to vector policies.

## 12.4 Example of supermodularity: Qs in tandem

- A set of queues in tandem. Each queue has a server whose speed is controlled.
- The utility of each server rewards the througput and penalizes the delay.
- The players then have compatible incentives: if one speeds up, the other also want to speed up.

•Consider two queues in tandem with i.i.d. exponentially distributed service times with parameters $\mu_i$, $i = 1, 2$. Let $\mu_i \leq u$ for some constant $u$.

•Server one has an infinite source of input jobs

•There is an infinite buffer between server 1 and 2.

•The throughput is given by $\mu_1 \wedge \mu_2$.

•The expected number of jobs in the buffer is given by

$$\frac{\mu_1}{\mu_2 - \mu_1}$$

when $\mu_1 < \mu_2$, and is otherwize infinite.

•Let

- $p_i(\mu_1 \wedge \mu_2)$ be the profit of server $i$,
- $c_i(\mu_i)$ be the operating cost,
- $g(\cdot)$ be the inventory cost.

•The utilities of the players are

$$f_1(\mu_1, \mu_2) := p_1(\mu_1 \wedge \mu_2) - c_1(\mu_1) - g\left(\frac{\mu_1}{\mu_2 - \mu_1}\right)$$

$$f_2(\mu_1, \mu_2) := p_2(\mu_1 \wedge \mu_2) - c_2(\mu_2) - g\left(\frac{\mu_1}{\mu_2 - \mu_1}\right).$$

•The strategy spaces are

$$S_1(\mu_2) = \{\mu_1 : 0 \leq \mu_1 \leq \mu_2\},$$

$$S_2(\mu_1) = \{\mu_2 : \mu_1 \leq \mu_2 \leq u\}.$$

•If $g$ is convex increasing then $f_i$ are supermodular.

## 12.5 Example of submodularity: Flow Control

- There is a single queueing centre
- The rates of two input streams to the queueing centre are controlled by 2 players.
- Similar utilities as before.

## 12.6 More detailed example:

•Consider two input streams with Poisson arrivals with rates $\lambda_1$ and $\lambda_2$.

•The queueing center consists of $c$ servers and no buffers. Each server has one unit of service rate.

•When all servers are occupied, an arrival is blcoked and lost.

•The blocking probability is given by the Erlang loss formula:

$$B(\lambda) = \frac{\lambda^c}{\lambda!} \left[ \sum_{k=0}^{c} \frac{\lambda^k}{k!} \right]^{-1}$$

where $\lambda = \lambda_1 + \lambda_2$.

•Suppose user $i$ maximimizes

$$f_i = r_i(\lambda_i) - c_i(\lambda B(\lambda)).$$

$c_i$ is assumed to be convex increasing.

$\lambda B(\lambda)$ is the total loss rate.

Then $f_i$ are submodular.

•Strategies: $\lambda_i \leq \overline{\lambda}$.

•Alternatively: $\lambda \leq \overline{\lambda}$. Then $S_i$ satisfy the descending property.