Connectivity and capacity in multi-hop wireless networks

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Wireless multi-hop networks

We consider wireless multi-hop networks where the nodes are randomly placed. Then, the network is a big random object, whose properties must be characterized.

- Connectivity between the nodes is a special kind of random graphs called random geometric graphs.
- The transmission medium cannot only be measured in bandwidth and time, but also in terms of space.

When the number of nodes in the network is large, some characteristic values become deterministic.
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Outline

1 Connectivity
- Modeling the node distribution
- Generalities
- Boolean connectivity
- Signal-to-interference ratio graph
- Rate based connectivity
- Nearest neighbors connectivity graph

2 Capacity
- Upper bounds: the concept of transport capacity
- Lower bounds: constructive schemes
Outline

1. Connectivity
   - Modeling the node distribution
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Node distribution

Two models are commonly used for generating the nodes’ location.

- I.i.d. distribution of a fixed number of nodes
- Poisson distribution

Both of these models are based on the fundamental assumption that nodes do not interact when choosing their position. In the sequel, we will assume in both case that nodes are uniformly distributed.
The uniform distribution is characterized by the two following properties:

1. The location of the nodes is i.i.d
2. The probability that node $i$ falls in a sub-area $B \in A$ is $|B|/|A|$.
Uniform distribution (properties)

The number of nodes in a sub-area $B$ is given by

$$\mathbb{P}(N(B) = m) = \binom{n}{m} \left( \frac{|B|}{|A|} \right)^m \left( 1 - \frac{|B|}{|A|} \right)^{(n-m)}$$

(binomial)

However, the number of nodes in two disjoint sub-areas is (negatively) **correlated**. Indeed, if we know that a node is in one sub-area, we also know that it is not in the other sub-area.

The node density is clearly

$$\lambda = \frac{n}{|A|}.$$
The Poisson distribution is characterized by the following properties:

1. The number of nodes in any collection of disjoint sub-areas is independent.
2. The number of nodes in a sub-area $B \in A$ follows a Poisson law:

$$
\mathbb{P}(N(B) = m) = \frac{(\lambda|B|)^m}{m!} e^{-\lambda|B|}.
$$
Poisson distribution (properties)

The total number of nodes in the total area is a random variable:

\[ P(N(A) = n) = \frac{(\lambda |A|)^n}{n!} e^{-\lambda |A|}. \]

(therefore, the actual node density is also random) However, the process is ergodic, and thus the actual node density tends to \( \lambda \) when the area tends to infinity.
Consider a sequence \( \{A_i\}_{i \in \mathbb{N}} \) of nested subsets of \( \mathbb{R}^d \) \((A_i \subset A_j \text{ if } i < j)\) so that

\[
\lim_{i \to \infty} |A_i| = \infty.
\]

For each \( i \), we assume a uniform distribution of \( \lfloor \lambda |A_i| \rfloor \) nodes.

**Property**

*For any sub-area \( B \in A_1 \), we have*

\[
\lim_{i \to \infty} \mathbb{P}(N(B) = m) = \frac{(\lambda |B|)^m}{m!} e^{-\lambda |B|}.
\]
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Connectivity: generalities

In a wireless network, we consider two nodes connected if somehow they can exchange data. The set of all connected pairs forms the *connectivity graph*. However, many parameters determine the presence of a link:

- Emitting power
- Path loss (distance)
- Background noise
- Interference from other transmissions
- Transmission rate
Connectivity: generalities

- However, to determine these parameters, one already has to make assumptions on the transmission strategy.
- Indeed, theoretically, every pair of nodes can exchange data at some very slow rate.
- Moreover, interferences can be sometimes canceled or even used constructively.

We will have to make restrictive assumptions to say something.
Boolean connectivity

The simplest assumption set is to:
- Neglect interferences
- Assume a deterministic path loss
- Assume a deterministic noise level
- Fix the transmission rate
- Fix the emitting power

We get the following condition:

\[
\frac{P\ell(d)}{N_0} \leq \beta.
\]

This condition only depends on the distance between source and destination. Therefore, this is equivalent to assigning a *transmission range*:

\[
r_{\text{xmit}} = \ell^{-1} \left( \frac{\beta N_0}{P} \right).
\]
One can improve the Boolean model by taking interferences into account. Assume that all nodes emit simultaneously with power $P$, the condition for successful transmission between $i$ and $j$ is:

$$\frac{P\ell(d_{ij})}{N_0 + \gamma \sum_{k \neq i,j} P\ell(d_{kj})}.$$
SINR connectivity

Properties:
- For $\gamma = 0$, this is a Boolean model
- The model is well defined only if
  \[ \int_0^\infty x^{d-1} \ell(x) dx < \infty \]  
  (Olber’s paradox!)
- The has less degrees of freedom than parameters.
We assume here that two nodes are connected if *somehow* they can exchange data at a given rate $R$ (which is a parameter of the model).

This model is very vague, but is intended to derive very general results rather than tight bounds.

Upper bounds in this model have to be of information theoretic nature, in order to capture every possible transmission scheme.
We notice that all models admit the transmission rate as a parameter:

- In the Boolean and SINR models: the SINR (SNR) threshold determines the rate of the transmission
- In the last model, the rate plays an explicit role

This reflects the fundamental connectivity-throughput trade-off:

- Either we have few high-capacity links
- or we choose to have many low-capacity links
In this model every node is connected to its $k$ nearest neighbors.

- Allows arbitrarily long edges
- Motivated by algorithmic concerns (power allocation)
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The Boolean model

We define the connectivity range as

\[ r_{tx} = \ell^{-1} \left( \frac{\beta N_0}{P} \right) \]

The same connectivity is obtained by centering a disk of radius \( r = r_{tx}/2 \) on each node.

The union of the disks forms the random set \( B(\lambda, r) \).
The Boolean model can also serve as a model for coverage (widely used for sensor networks). However, coverage is an easy problem compared to connectivity.

**Proposition**

The fraction of the space covered by a disk is given by

\[ 1 - \exp(-\lambda \pi r^2) \]

Proof: compute the probability that there is no node is a disk of radius \( r \).
Full coverage

As an example, we compute the asymptotic range necessary for full coverage.

**Assumptions**

- Constant area $A$ (with $|A| = 1$)
- Growing number $n$ of uniformly distributed nodes
- $r$ is a function of $n$ (to determine).

We need to choose $r$ so that

$$\lim_{n \to \infty} 1 - \exp(-\lambda \pi r^2) = 1.$$  

(we say that the area is fully covered *with high probability*)
The node density is $n$ the node distribution tends to a Poisson distribution.

The law is thus

$$r = \frac{\omega(n)}{\sqrt{n}}$$

where $\omega(n)$ is a function such that

$$\lim_{n \to \infty} \omega(n) = \infty.$$
Full connectivity (lower bound)

To get a lower bound on the critical radius for full connectivity, we can use the following argument:

\[ P(\text{The network is disconnected}) \geq P(\text{Some node is isolated}) \]

For each node \( i \) we have

\[ P(i \text{ is the only isolated node}) \]
\[ \geq P(i \text{ is isolated}) - \sum_{j \neq i} P(i \text{ and } j \text{ are isolated}) \]
\[ P(i \text{ is isolated}) - (n - 1)P(i \text{ and } j \text{ are isolated}) \]
Full connectivity (lower bound)

Thus

\[ P(\text{Some node is isolated}) \geq P(\text{Exactly one node is isolated}) \]

\[ = \sum_{i=1}^{n} P(i \text{ is the only isolated node}) \geq n (P(i \text{ isolated}) - (n - 1)P(i \text{ and j isolated})) \]

We get the condition:

**Theorem (Gupta & Kumar, 1999)**

\[ r = \frac{\log n + \omega(n)}{4n} \]
The upper bound is much more complicated to derive, as the fact that no node is isolated does not imply that the network is connected. However, intuitively, we can see that it is asymptotically the case:

- The average node degree increases to infinity (like \( \log n \))
- For any fixed \( k \), nodes have less than \( k \) neighbors with vanishing probability

Penrose proved that it is actually the case:

**Theorem (M. D. Penrose, 1999)**

Let \( r_n \) (resp. \( r'_n \) be the critical radius for full connectivity (resp. minimum node degree larger than one). Then

\[
\lim_{n \to \infty} \mathbb{P}(r_n = r'_n) = 1.
\]
However, we still need to compute an upper bound to the critical radius $r'_n$.

$$\mathbb{P}(i \text{ is isolated}) = \exp(-4\lambda \pi r^2)$$

and thus

$$\mathbb{P}(\text{No node is isolated}) \geq 1 - n \exp(-4n\pi r^2)$$

Therefore we have

$$1 - n \exp(-4n\pi r^2) \rightarrow 1$$

$$\frac{n}{\exp(4n\pi r^2)} \rightarrow 0$$

$$\log n - 4n\pi r^2 \rightarrow -\infty$$

$$\pi r^2 - \frac{\log n}{4n} \rightarrow \infty$$

$$\pi r^2 = \frac{\log n}{4n} + \omega(n)$$
Extended networks

The same result exists for the case where the density is constant and the area goes to infinity:

**Theorem**

Let the area be the square \( A = [0, \sqrt{n}] \times [0, \sqrt{n}] \) with \( n \) uniformly placed nodes. Then the network is fully connected w.h.p. iff

\[
    r = \frac{\log n}{4} + \omega(n).
\]

It is easy to switch from *dense* networks to *extended* networks by rescaling the whole thing by a factor \( \sqrt{n} \). Note that the average node degree \( N = 4\pi r^2 \) remains constant.
Partial connectivity (intro)

The full connectivity (resp. full coverage) approach suffers from a few drawbacks:

- Quite artificial $n \to \infty$
- Even more artificial $r \to 0$ (resp. $\infty$)

Furthermore, *full* connectivity is not always required. A good 99% connectivity would be OK in most cases.
Partial connectivity (setting)

The idea is to compute the critical radius for a given connectivity factor \( 0 < \theta < 1 \) (\( \theta \)-connectivity).

- We consider an infinite area.
- To get \( \theta \)-connectivity, we need a connected component in the network that contains a fraction \( \theta \) of the nodes.
- This component contains thus an infinite number of nodes.

The existence of such an infinite component is the called percolation.
Percolation

Infinite number of finite clusters

A big connected component and finite clusters

A fully connected network
Percolation

Definition

The percolation function $\theta(\lambda)$ is the probability that the origin belongs to an unbounded component of $B(\lambda, r)$.

Main result of percolation for the Boolean model:

Theorem

Given $r$, there exists a critical density $0 < \lambda_c < \infty$ such that if $\lambda > \lambda_c$, $\mathbb{P}(\ell(W) = \infty) > 0$, where $W$ denotes the component where the origin lies ($= \emptyset$ if the origin is not covered)
Percolation functions

There are two functions characterizing the degree of connectivity of the model:

- The standard percolation function:
  \[ \theta(\lambda) = \mathbb{P}(\ell(W(\{0\})) = \infty) \]

- An alternative percolation function
  \[ \theta'(\lambda) = \mathbb{P}(\ell(W(S(0, r))) = \infty) \]

where \( W(A) \) denotes the union of the connected components that intersect \( A \) (thus \( W(\{0\}) = W \)). Note that \( \theta' \) is also the probability that the origin is in an unbounded component **given that there is a node at the origin** (Palm probability).
Percolation function

A plot of $\theta'$ (with $N = 4\pi r^2$):
Proof of the non-triviality of the threshold

We have to show two things (for a given $r$):

- There exists a density $\lambda < \infty$ for which $\theta(\lambda) > 0$,
- There exists a density $\lambda > 0$ for which $\theta(\lambda) = 0$.

We ignore the following additional (interesting) questions:

- Unicity of the giant cluster,
- Equivalence with other definitions of the critical threshold.
The first idea is to map the Boolean model onto a discrete model:

We pick $c = \frac{r}{\sqrt{2}}$, so that balls centered on neighboring squares overlap.
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Mapping

For each realization of the Poisson point process, we get a realization of a bond percolation process:

\[ \Phi : \mathbb{R}^2 \longrightarrow \{0, 1\}^\mathbb{L}^2 \]

\[ X \longrightarrow L \]

The probability measure on \( \{0, 1\}^\mathbb{L}^2 \) is a product measure (i.i.d. process). Denoting by \( p \) the probability that an edge is open, we get

\[ p = \mathbb{P}(X(\text{square}) \geq 1) = 1 - \exp(-\lambda d^2) \]

and we have

\[ \{L \text{ contains an unbounded component}\} \]
\[ \Rightarrow \{X \text{ contains an unbounded component}\} \]
We observe that in $\mathbb{L}^2$, the origin belong to a finite component if and only if it is surrounded by a closed circuit in the dual lattice.
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Peierl’s argument

- A circuit of length $2k$ is closed with probability $(1 - p)^{2k}$ (there is no circuit with an odd number of edges).
- There are at most $3^{2k-1}k$ circuits surrounding the origin.
The probability that there exists a closed circuit around the origin is thus smaller than

\[ \sum_{k=2}^{\infty} 3^{2k-1} k (1 - p)^{2k} \]

This sum converges and is strictly smaller than one for \( p \) large enough (i.e. \( \lambda \) large enough).
Non-existence

We start with another mapping:

We want to show that the origin is surrounded by such a circuit with probability one.
Site percolation

Here, we map the Poisson process onto a site percolation process:

\[ \Phi : \mathbb{R}^2 \rightarrow \{0, 1\}^{\mathbb{N}^2} \]
\[ X \rightarrow S \]

where the probability measure on \( \{0, 1\}^{\mathbb{N}^2} \) is a product measure and

\[ \{X \text{ contains an unbounded component}\} \]
\[ \Rightarrow \{S \text{ contains an unbounded component}\} \]
Bounded components

In the site percolation model, there exists a closed circuit if and only if the component of the origin is bounded in the following graph:
To compute an upper bound on the probability that the cluster at the origin is bounded, we compare the construction of this cluster to a *branching process*:

- We start from the origin. It has 9 neighbors.
- Among these neighbors, some are open with probability $p$, so that we add in average $9p$ new nodes in the cluster.
- Each of these $9p$ nodes have at most 9 neighbors, and thus less than $9p$ descendant in average.
- The branching process dies with probability one is $p < 1/9$.

As $p = 1 - \exp(-\lambda d^2)$, we get a threshold for $\lambda$. 
We thus proved that:

- If $\lambda$ is large enough, we are not sure to find a closed circuit around the origin.
- If $\lambda$ is small enough, the construction of the component at the origin dies with probability one.

This shows the non-triviality of the percolation threshold in the Boolean model.
Monotonicity of the percolation function

We show now that the function $\theta$ (resp. $\theta'$) is increasing. To do that, we use a coupling technique:

- Let us consider two Poisson point processes on $\mathbb{R}^2$, $X$ and $X_\varepsilon$ of density $\lambda$ and $\varepsilon$ respectively.
- The union of the two processes is also a Poisson point process, $X'$, of density $\lambda + \varepsilon$.
- If the origin belongs to an unbounded component in $B(\lambda, r)$, so does it in $B(\lambda', r)$. 
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- If the origin belongs to an unbounded component in $B(\lambda, r)$, so does it in $B(\lambda', r)$. 
All of this is true for infinite networks. What about finite domains?

**Theorem (Penrose - Pisztora 1996)**

Let $B(m)$ denote the square $[0, m] \times [0, m]$, and let $C_\infty$ denote the largest cluster in $B(m)$. Then there exists $c > 0$ such that

$$
P \left( 1 - \varepsilon \leq \frac{\ell(C_\infty)}{m^2 \theta(\lambda, r)} \leq 1 + \varepsilon \right) \geq 1 - \exp(-cm).$$
The following property is also very useful:

Property

Let $\lambda > \lambda_c$ and let $R(m)$ denote the rectangle $[0, m] \times [0, \alpha m]$ for some fixed $\alpha$. Then

$$\lim_{m \to \infty} \mathbb{P}(\text{There is a left-right crossing in } R(m)) = 1$$
FKG inequality

Definition (Increasing event)

Let \( X \) be a realization of the Poisson point process, and \( X' \) another realization that contains all the points in \( X \) plus a few more. An event \( E \) is said increasing if \( 1_{E(X)} \leq 1_{E(X')} \) for all such \( X \) and \( X' \).

Property (FKG)

Let \( A \) and \( B \) be two increasing (resp. decreasing) events. Then

\[
P(A \cap B) \geq P(A) P(B)
\]

Note: the existence of crossings are increasing events. So is the existence of an unbounded cluster.
In 1D, the Boolean model can be seen as a $M/D/\infty$ queue, which is always stable. This implies that there is no percolation in 1D.

One can compute bounds on the probability that two points are connected in a 1D Boolean model given the distance $x$ between them:

**Theorem**

$$P_c(x) \leq \left(1 - e^{-\lambda r}\right) e^{-\lambda(x-r)} e^{-\lambda r}.$$

This bound decreases exponentially with $x$. 

$$(1 - e^{-\lambda r}) e^{-\lambda(x-2r)} e^{-\lambda r} - \lambda e^{-\lambda r} \leq$$
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   - Upper bounds: the concept of transport capacity
   - Lower bounds: constructive schemes
In the Boolean model, we considered $\lambda$ as the only parameter. In this model, we add the *interference weighting factor* $\gamma$ as a second parameter. All other parameters are supposed constant.

\[
\frac{P\ell(d_{ij})}{N_0 + \gamma \sum_{k \neq i,j} P\ell(d_{kj})} \geq \beta.
\]
We know that if $\gamma = 0$, the model is a Boolean model. Let us denote by $\lambda_c$ the critical density for that Boolean model. As increasing $\gamma$ only degrades connectivity (this can be shown using a coupling argument), no percolation occurs if $\lambda < \lambda_c$, whatever the value of $\gamma$ is.

For any $\gamma > 0$, the nodes’ degree is uniformly bounded by

$$\#(\text{Neighbors}) \leq \left\lfloor \frac{1}{\gamma \beta} \right\rfloor \quad \text{(pole capacity)}$$

Thus, if $\gamma \leq 1/\beta$, then no node can have more than one neighbor, and no percolation occurs.
Open questions

- Is there any percolation for $\gamma > 0$? For what values of $\lambda$?

**Theorem**

For any $\lambda > \lambda_c$, there exists $\gamma_c(\lambda) > 0$ such that percolation occurs if $\gamma < \gamma_c$.

- Note that we know already that $\gamma_c(\lambda) < 1/\beta$. 
Percolation in the SINR model

- The case $\gamma = 0$ correspond to a Boolean model with ball radius $\ell^{-1}(\beta N_0 / P)$, with critical density $\lambda_c$.
- We consider a Boolean model with $\lambda > \lambda_c$.
- As $\lambda > \lambda_c$, we can find a radius $r < \ell^{-1}(\beta N_0 / P)$ such that the model still percolates.
- We take $B(\lambda, r)$ as the starting point of the proof.
We construct the following mapping on $\mathcal{B}(\lambda, r)$:

Here, we use Property 2 and the FKG inequality to ensure that the probability that such a crossing has a high probability: This technique is called renormalization.
Open edges

We declare an edge $e$ open if the event $C_e = A_e \cap B_e$ occurs:

- $A_e$: There exist crossings as shown in the previous slide.
- $B_e$: The value of the *shot-noise process* does not exceed a threshold $M$ in the whole rectangle.

The shot-noise process at $y$ is defined as

$$l(y) = \sum_{x \in X, x \neq y} \ell(|x - y|)$$
Shot noise
In order to play Peierl’s argument, we need that the sum over all possible closed circuits converge somehow. Remember that the sum looked like:

$$\sum_{k=2}^{\infty} 3^{2k-1} k \mathbb{P}(\text{circuit of length } 2k \text{ is closed})$$

Here, the question is to evaluate

$$\mathbb{P}(\text{circuit of length } 2k \text{ is closed})$$.
Events A

For the events of type $A$, the problem is simple:

- every non-adjacent edge is independent
- the set of every second edge in a circuit contains only non-adjacent edges
- the probability that all edges are closed is less than the probability that every second edge is closed.

We get:

$$\Pr(\text{circuit of length } 2k \text{ is closed}) \leq (1 - p_A)^k$$

where $p_A$ is the probability that an event $A_e$ occurs.
We start with a reformulation of the definition of Event $B$:

- We define the shifted attenuation function as follows:
  \[
  \ell'(x) = \begin{cases} 
  \ell(x - \sqrt{10d/4}) & \text{if } x > \sqrt{10d/4} \\
  0 & \text{otherwise}
  \end{cases}
  \]

- The event $B_e$ occurs if the shot noise 
  \[
  I'(y) = \sum \ell'(|x - y|)
  \]
  is less than $M$ at the center of Edge $e$. 
Using a generalized Chernov bound, we have, for a sequence of edges $e_1, e_2, \ldots, e_{2k}$,

$$\mathbb{P}(\bar{B}_{e_1} \cap \bar{B}_{e_2} \cap \cdots \cap \bar{B}_{e_{2k}}) \leq e^{-2ksM} \mathbb{E}\left( e^{s \sum_{i=1}^{2k} l'(x_{e_i})} \right)$$

where $s$ is an arbitrary constant and $x_{e_i}$ the center of Edge $e_i$.

By bounding the right-hand side, one can obtain the desired product form.
Campbell’s Theorem

\[ \mathbb{E} \left( e^{s \sum_{x \in X} f(x)} \right) = \exp \left( \lambda \int_{\mathbb{R}^2} (e^{sf(x)} - 1) \, dx \right) \]
Taking

\[ f(x) = \sum_{i=1}^{2k} \ell'(|x - x_{e_i}|) \]

in Campbell’s theorem, we get

\[
\mathbb{E}\left( e^{s \sum_{i=1}^{2k} l'(x_{e_i})} \right) = \exp \left( \lambda \int_{\mathbb{R}^2} (e^{s \sum_{i=1}^{2k} \ell'(|x-x_{e_i}|)} - 1) dx \right).
\]

Therefore, one can find \( p_B \) such that

\[
P(\bar{B}_{e_1} \cap \bar{B}_{e_2} \cap \cdots \cap \bar{B}_{e_{2k}}) \leq (1 - p_B)^k.
\]
Combining events A and B

Remember that an edge $e$ is open only if events $A_e$ and $B_e$ occur. Using Schwartz inequality, we can combine the two previous results and obtain the inequality

$$\mathbb{P}(\bar{C}_{e_1} \cap \bar{C}_{e_2} \cap \cdots \cap \bar{C}_{e_{2k}}) \leq q^k$$

for some $q < 1$, where $C_e = A_e \cap B_e$. The existence of a percolation phenomenon follows from Peierl’s argument.
Domination by product measures

This result can also be obtained using a more general result:

**Theorem (Liggett, Schonmann and Stacey, 1997)**

Let $G = (V, E)$ be a graph with bounded degree, and $\{A_x\}$ a family of events indexed by the vertices $x \in V$. If there exists $p$ large enough such that

$$
\mathbb{E}(1_{A_x} | A_y, xy \notin E) \geq p \quad \text{a.s.}
$$

then the probability measure of the events $\{A_x\}$ dominates a product measure with positive density $p'$. Moreover, $\lim_{p \to 1} p' = 1$. 
Simulation

Percolation domain

Node density vs. $\gamma^*$

- Power law
- Truncated power law

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Capacity
- Upper bounds
- Lower bounds
Power law attenuation function

When the attenuation function has the form

$$\ell(x) = cx^{-\alpha},$$

the model acquires an artificial scaling property:

$$\frac{P_\ell(cd_{ij})}{N_0 + \gamma \sum_{k \in \text{Intf}} P_\ell(cd_{kj})} = \frac{P_\ell(d_{ij})}{c^\alpha N_0 + \gamma \sum_{k \in \text{Intf}} P_\ell(d_{kj})}$$

Thus, increasing the node density can only help, and $\gamma^*(\lambda)$ is an increasing function. If $N_0 = 0$, the model is scale free.
For a bounded attenuation function $\ell(x)$, when the density $\lambda$ increases:

$$\frac{\text{bounded} \left\{ P\ell(\|X_i - X_j\|) \right\}}{N_0 + \gamma \sum_{k\neq i,j} P\ell(\|X_k - X_j\|)}$$

unbounded
Graphical representation of the bounds

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Node distribution
Generalities
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SINR
Rate based
Nearest neighbors

Capacity
Upper bounds
Lower bounds

$\lambda^*$

$1/\beta$

$\sim 1/\lambda$

super–critical region

$\sim 1/\lambda$
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- Upper bounds: the concept of transport capacity
- Lower bounds: constructive schemes
Rate based connectivity

In this model, we will cover the two approaches:

- Full connectivity
- Partial connectivity

In both cases, we can only compute bounds on the critical rate.

- The upper bound is computed with an information theoretic capacity argument
- The lower bound is computed using a constructive scheme
We consider the case where a node emits, and all other nodes in the network collaborate to receive.

- The total power received by the other nodes is $P$ times the value of the shot noise at the location of the emitter.

$$P_{\text{tot}} = P \sum_{x \in X, x \neq y} \ell(|x - y|)$$

- The maximum rate achievable from this emitter to the other nodes is bounded by

$$R \leq \frac{1}{2} \log \left(1 + \frac{P_{\text{tot}}}{N_0}\right)$$
All we need to do is to compute the fraction of nodes that are such that $P_{\text{tot}}$ is high enough to have a chance to achieve $R$.

- In an infinite network, by ergodicity, the fraction of such nodes is equal to the probability that the shot noise exceeds the threshold value.
- In a finite (extended) network with $n \to \infty$, $R$ has to go to zero to ensure that this fraction goes to one. One can show that for $\ell(x) \sim x^{-\alpha}$, if

$$R \geq \frac{c_1}{(\log n)^{\alpha-1}}$$

then there exists a node that cannot achieve $R$ w.h.p.
If one finds a path between source and destination where the longest hop has length $2r$, then the rate

$$R = \frac{1}{8} \log \left(1 + \frac{P\ell(2r)}{N_0 + P \sum_{k=0}^{\infty} 6k\ell(2kr)}\right)$$

is achievable.
In an infinite network, one finds such a path if source and destination belong to the same component in the Boolean model.

In a finite (extended) network with \( n \to \infty \), one has to increase the maximum hop length like \( \log n \) to keep all nodes connected, and thus decrease \( R \). For \( \ell(x) \sim x^{-\alpha} \), if

\[
R \leq \frac{1}{8} \log \left( 1 + \frac{c_2}{(\log n)^\alpha} \right),
\]

the network is connected.
Bounds

Connectivity and capacity in multi-hop wireless networks

O. Dousse

Connectivity
Node distribution
Generalities
Boolean
SINR
Rate based
Nearest neighbors

Capacity
Upper bounds
Lower bounds

<table>
<thead>
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<th>Rate R</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fraction of nodes

Rate R

Lower bound
Upper bound
Outline

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   - Upper bounds: the concept of transport capacity
   - Lower bounds: constructive schemes
Nearest neighbors: partial connectivity

- Lower bound: if nodes connect only to one neighbor, all components have size two, and no percolation occurs.
- Upper bound: a simple mapping (surprise?) can give an upper bound of 64
We have if the average number of node per square is one,

\[ P(N(\text{square}) \in \{1, 2\}) > \frac{1}{2} \]

Therefore, more than a half of the edges are declared open.
Fixed degree: full connectivity

Theorem (Xue Kumar 2004)

*The number of neighbor must scale like log n to ensure full connectivity.*

The proof proceeds by dividing the area into squares of size log n. It is easy to show that each square contains between $(1 - \varepsilon) \log n$ and $(1 + \varepsilon) \log n$ nodes. An argument similar as the one for partial connectivity allows to compute the upper bound.
The lower bound can be obtained by studying the “$k$-filling events” in the little squares:

$$\sqrt{\log n}$$
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Extended vs. dense networks

In dense networks:

- With power law attenuation functions, the system scales well when the node density increases to infinity.
- However, physics requires that the attenuation function is bounded.
- This is the problem of near field effect.

Such problems do not occur in extended networks. Therefore, the two approaches must be treated separately.
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The concept of transport capacity

When multiple connections occur simultaneously in the network, one uses the concept of transport capacity to derive upper bounds.

Definition

The transport capacity of the network is the number of bits per unit of time and per unit of distance that can be forwarded.

This definition captures the fact that long distance communication consumes more resource than short links:

- Multi-hopping implies several emitters for a single flow
- Direct long links require a large silent area around them to be successful
Protocol Model

Definition

A connection is successful if the receiver is close enough compared to the closest interferer.

- This leads to the concept of *footprint* of a link.
- The rate of all links is the same.
Gupta & Kumar bound

In a dense network:

- Each link of length $d$ has a footprint of size of order $d^2$.
- A link of length $d$ contributes to the transport capacity by $d$.
- One can pack at most $1/d^2$ active links of length $d$.
- The total transport capacity is $1/d$.
- The typical distance between nodes decreases like $1/\sqrt{n}$.
- The transport capacity scales like $\sqrt{n}$.

The same results can be adapted to the extended network case (transport capacity scales like $n$).
Application of the result

- **Uniform traffic matrix**
  - Typical flow length is of order 1
  - There are $n$ flows
  - The throughput per flow is $1/\sqrt{n}$

- **Hybrid networks (Liu & Towsley)**
  - One adds $\sqrt{n}$ base stations.
  - The flows have length $1/\sqrt{n}$
  - The throughput per flow remains constant
Application of the result (cntd.)

- Sparse traffic matrix
  - Of course, if the number of flows grows like $\sqrt{n}$, the throughput scales.

- Local communications
  - If the distance between source and destination decreases like $1/\sqrt{n}$, the throughput also remains constant.
Physical model

- In this model, the cumulative interference is taken into account (in the protocol model, only the distance to the first interferer matters).

\[
\frac{P_\ell(cd_{ij})}{N_0 + \gamma \sum_{k \in \text{Intf}} P_\ell(cd_{kj})} > \beta
\]

- This is still a point to point communication model.
- The result on transport capacity remains the same for extended networks.
- If the node density increases, the behavior depends on the shape of \( \ell \) around zero.
High density regime

- If $\ell$ is a power law, then things scale well for high node densities, and dense networks behave like extended networks (scale free property).
- If the attenuation function is bounded, adding more nodes does not help the transport capacity, which saturates. The throughput decreases thus like $1/n!!$.
We consider a finite area $A$, and thus there exists

$$m \doteq \min_{x, y \in A} \ell(|x - y|) > 0.$$ 

We know that the contribution of a link from node $i$ to node $r(i)$ to the transport capacity is

$$\delta_i \frac{P_i \ell(\delta_i)}{N_0 + \sum_{k \neq i, r(i)} \ell(|x_k - x_{r(i)}|)}$$

where $\delta_i = |x_i - x_{r(i)}|$. 

Dominant nodes

We pick a constant $0 < \varepsilon < 1/2$ and call a node $i$ dominant if its emitting power $P_i$ is at least $\varepsilon$ time the total power emitted in the network

$$P_{\text{tot}} \doteq \sum_i P_i.$$ 

There are at most $1/\varepsilon$ such nodes. Their contribution to the transport capacity as emitter or as receiver is at most

$$\frac{2}{\varepsilon} \max_{\delta} \delta P_{\text{max}} \ell(\delta) \frac{N_0}{N}.$$ 

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All the other nodes emit with a power less than $\varepsilon P_{tot}$. Their contribution is at most

$$\sum_{\delta_i \text{ non-dominant}} \delta_i \frac{P_i \ell(\delta_i)}{N_0 + \sum_{k \neq i, r(i)} P_i \ell(|x_k - x_{r(i)}|)}$$

$$\leq \sum_{\delta_i \text{ non-dominant}} \delta_i \frac{P_i \ell(\delta_i)}{mP_{tot}(1 - 2\varepsilon)}$$

$$\leq \max_\delta \frac{P_{tot} \ell(\delta)}{mP_{tot}(1 - 2\varepsilon)}$$
All of these bounds require quite strong assumptions on the technology:

- Links all have the same rate
- Interferences are treated as noise
- Only point to point communication

There is a simple argument that allows to recover a similar bound, but without any assumption on the communication technology:

- Divide the square into two equal rectangles
- One half of the flows cross the boundary between the rectangles
- Assume that all nodes on one side collaborate to transmit to the other side
- We add “mirror nodes” to make the gain matrix $L_{ij} = \ell(|x_i - y_j|)$ symmetric
Proof (Lévêque, Telatar)

Cut-set bound: the sum of the rates of the flows crossing the boundary is bounded by

\[ R \leq \max_{P_k \geq 0, \sum P_k \leq nP} \sum_{k=1}^{n} \log \left( 1 + \frac{P_k \lambda_k^2}{N_0} \right) \]

where \( \lambda_k \) is the \( k^{\text{th}} \) largest eigenvalue of the matrix \( LL^* \), \( L = \{ \ell(|x_i - x_j|) \} \). Then,

\[ R \leq \sum_{k=1}^{n} \log \left( 1 + \frac{nP \lambda_k^2}{N_0} \right) \]

\[ \leq 2 \sum_{k=1}^{n} \log \left( 1 + \frac{\sqrt{nP} \lambda_k}{N_0} \right) \]

\[ \leq 2 \sum_{k=1}^{n} \log \left( 1 + \frac{\sqrt{nP} L_{kk}}{N_0} \right) \]
Thus,

\[ R \leq \sum_{k=1}^{n} \log \left( 1 + \frac{\sqrt{n}P\ell(\hat{x}_k)}{N_0} \right). \]

By projecting on the horizontal line, this can be seen as a truncated shot-noise, with density \( \sqrt{n} \).

For each value of \( n \), we define the Poisson point process \( \Phi_n \) with intensity \( \sqrt{n} \) on \( \mathbb{R}^+ \). We get

\[ R \leq I \overset{\text{def}}{=} \sum_{x \in \Phi_n} \log \left( 1 + \frac{\sqrt{n}P\ell(\hat{x}_k)}{N_0} \right). \]
Proof (cntd.)

By Campbell’s theorem, the average of $I$ is

$$\mathbb{E}(I) = \sqrt{n} \int_0^\infty dx \log \left( 1 + \frac{\sqrt{nP\ell(\hat{x}_k)}}{N_0} \right)$$

which can be shown to be bounded by

$$\mathbb{E}(I) \leq K_1 n^{1/2 + 1/\alpha}$$

The variance reads

$$\text{Var}(I) = \sqrt{n} \int_0^\infty dx \log \left( 1 + \frac{\sqrt{nP\ell(\hat{x}_k)}}{N_0} \right)^2$$

which can be bounded by

$$\text{Var}(I) \leq K_2 n^{1/2 + 2/\alpha}$$
Using a simple Chebychev bound we get

$$P(I > (1 + \varepsilon)\mathbb{E}(I)) \leq \frac{\text{Var}(I)}{\varepsilon^2 \mathbb{E}(I)^2} \to 0$$

Therefore, with high probability,

$$R \leq (1 + \varepsilon)\mathbb{E}(I) \leq (1 + \varepsilon)K_1 n^{1/2 + 1/\alpha}$$

which has to be shared among the $n$ flows. The throughput per flow is thus of order

$$n^{1/\alpha - 1/2}$$
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Construction

- We consider a square of size $[0, \sqrt{n}] \times [0, \sqrt{n}]$ with unit node density.
- We choose a predefined hop length $2r$ such that the Boolean model with ball radius $r$ is super-critical.

**Percolation result**

The number of paths from the left border to the right border (crossings) of the network is proportional to $\sqrt{n}$. 
Construction

- Horizontal and vertical crossings thus form a “regular grid”
- Furthermore, a constant throughput is achievable simultaneously on all crossings, using simple time sharing.
Throughput under uniform traffic matrix

- Divide the network into squares of constant size, so that there is a one-to-one relation between the rows and the horizontal paths (resp. columns and vertical paths)
- The probability that all rows and columns contain no more than $\alpha \sqrt{n}$ nodes tends to one
- The paths do not deviate much from their slabs, so that the cost of draining the traffic from the actual location of the sources to the paths is of order log $n$. 
Throughput under uniform traffic matrix

How to achieve a $\Theta(1/\sqrt{n})$ throughput?

- Route the packets in a deterministic way along the square grid (for instance, horizontally first, and then vertically)
- The load on each path is of order $O(\sqrt{n})$
Draining

- However, the paths are not always close to the source (resp. destination).
- They can deviate by a distance of order $\log n$.
- The capacity needed to transport the data from their source to the paths (resp. from the paths to the destination) is much less than the capacity needed to transport the data along the paths.
- Draining the data to the paths is asymptotically negligible.
Thanks!
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Inter-Perf 2007

Workshop on interdisciplinary approaches in the modelling of computer & communication systems
Nantes, France, on October 26, 2007 (co-located with Valuetools 2007)

- http://www.inter-perf.org
- Deadline: May 20, 2007